PSEUDOSPECTRA AND OPTIMIZATION

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Technion: Lecture 2 **Joint work with J. Burke and M. Overton**

1. OUTLINE

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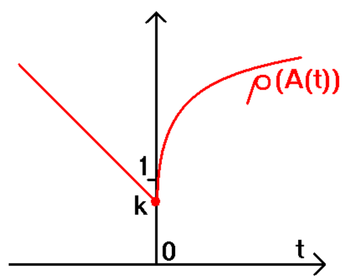

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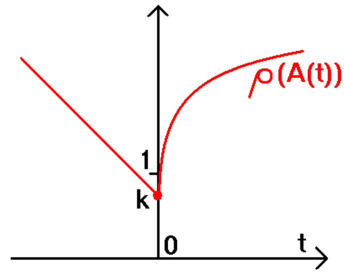

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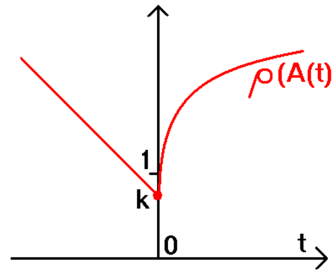

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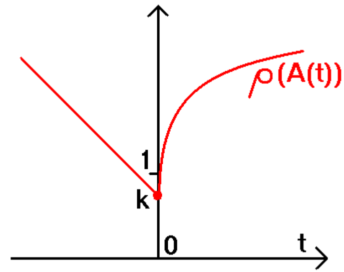

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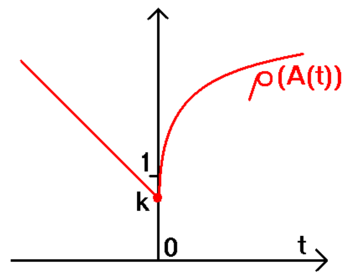

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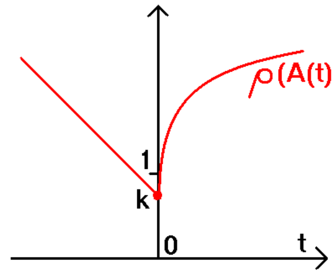

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Analogously, in **continuous time**, $e^{At} \to 0$ with peaks not too large when $\Lambda_\epsilon(A)$ lies in the left halfplane for ϵ not too small.

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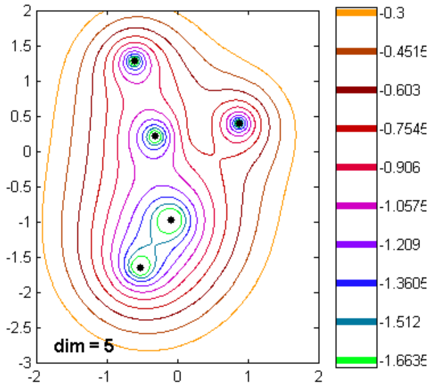

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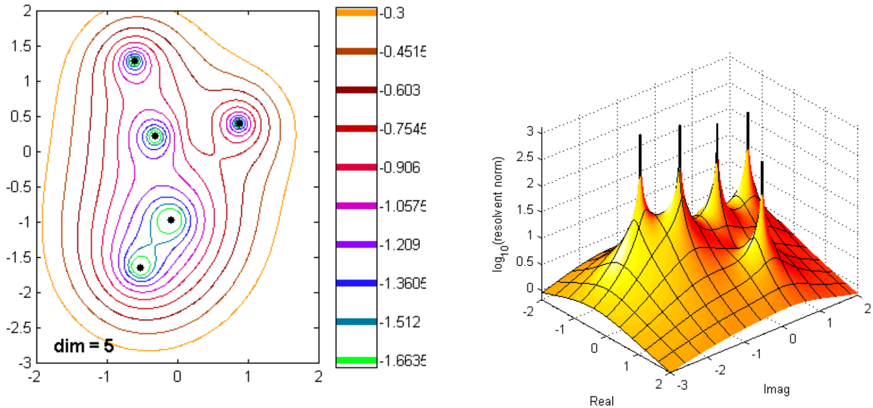

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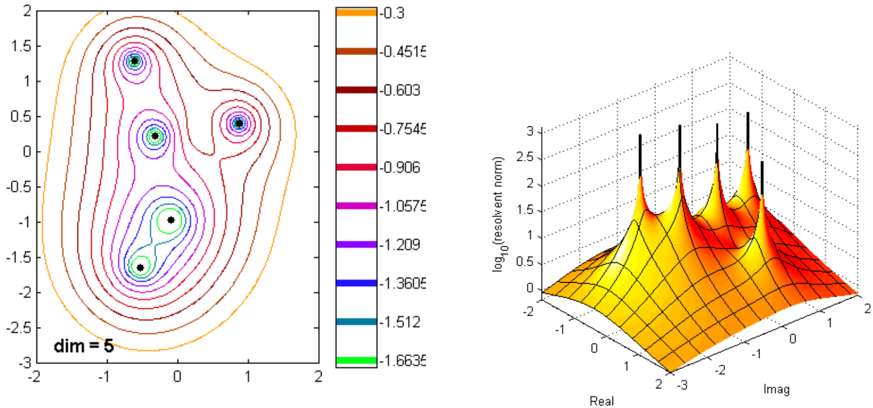

Demmel's example: $A = -\begin{bmatrix} 1 & 5 & 5^2 & 5^3 & 5^4 \\ 0 & 1 & 5 & 5^2 & 5^3 \\ 0 & 0 & 1 & 5 & 5^2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ldots$

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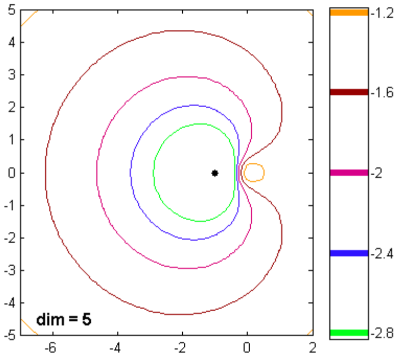

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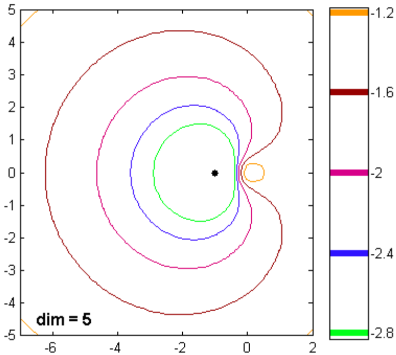

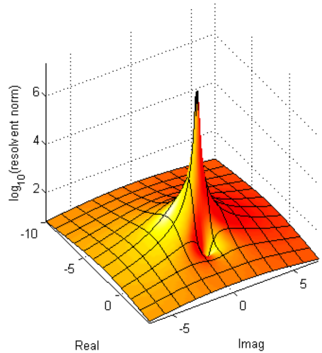

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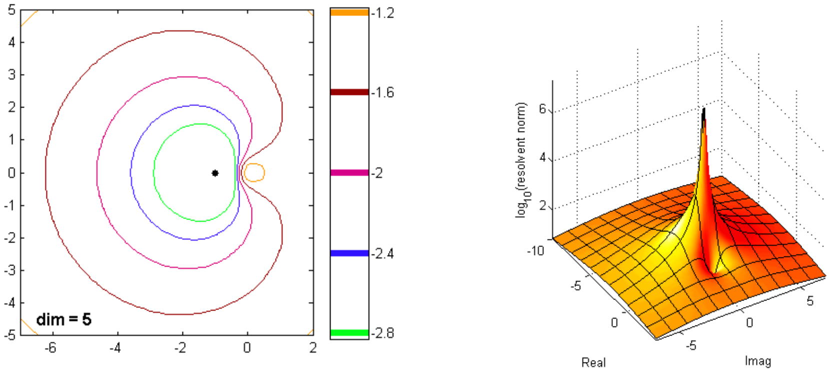

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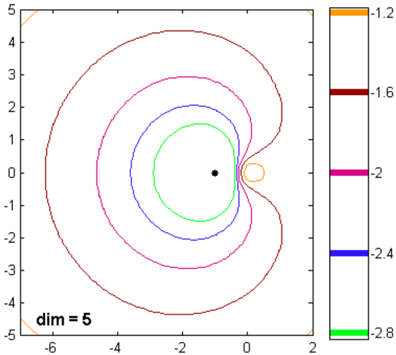

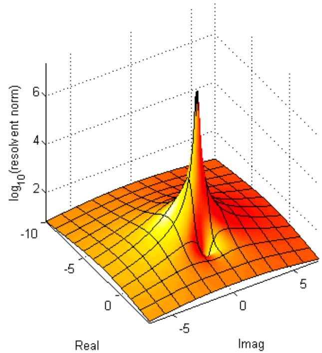

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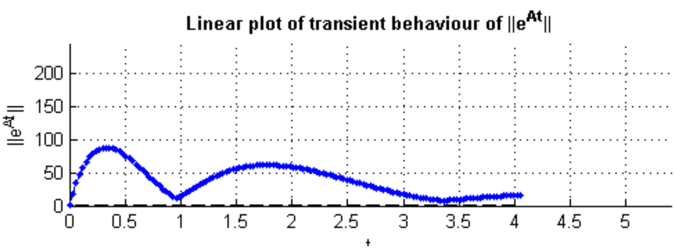

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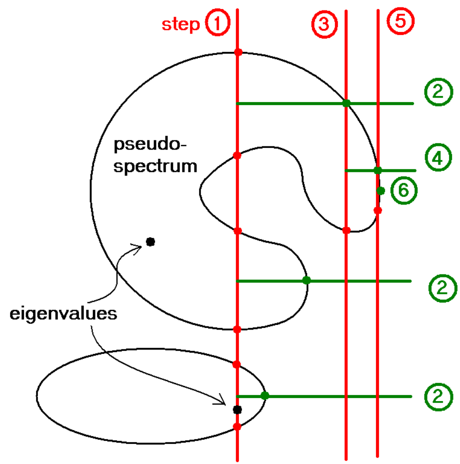

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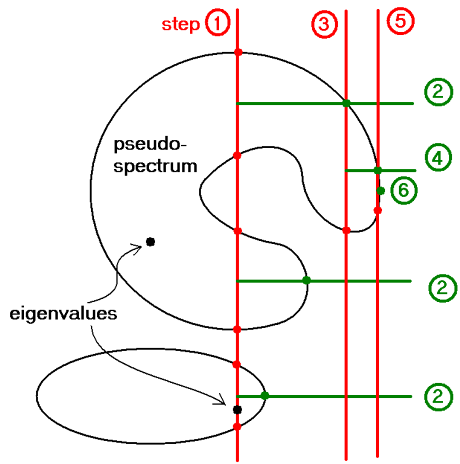

Hence a globally and quadratically convergent **criss-cross algorithm** for α_ϵ.

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Hence nonsmooth **gradient sampling** for **optimizing** α_ϵ.

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What about the pseudospectral map $A \mapsto \Lambda_\epsilon(A)$?

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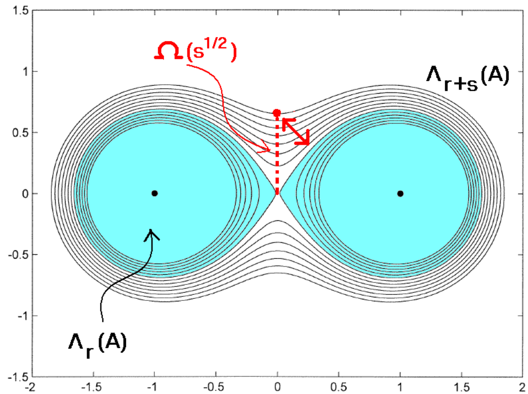

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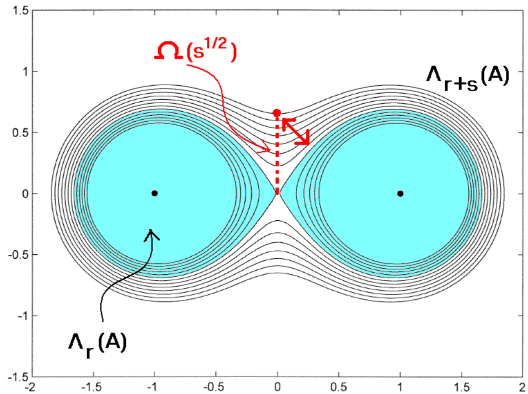

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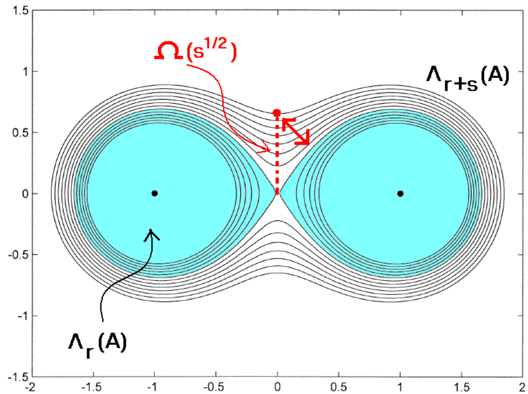

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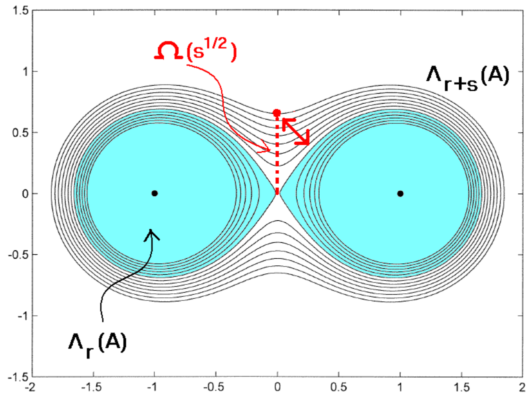

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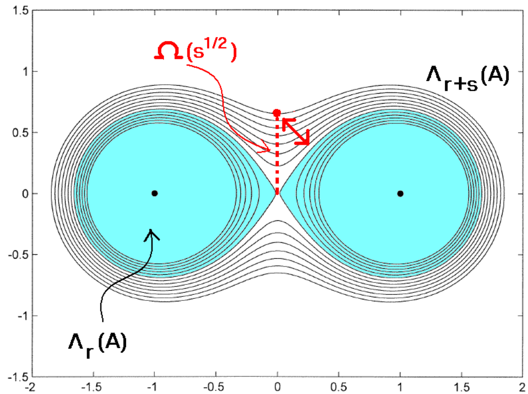

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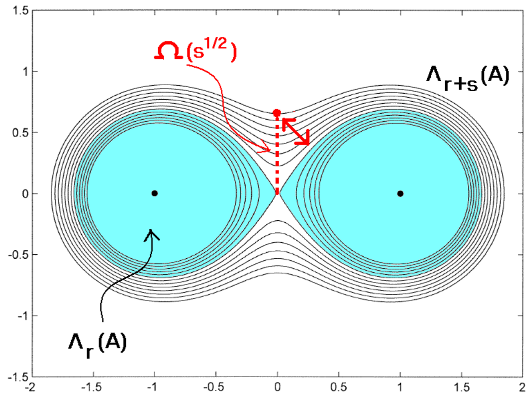

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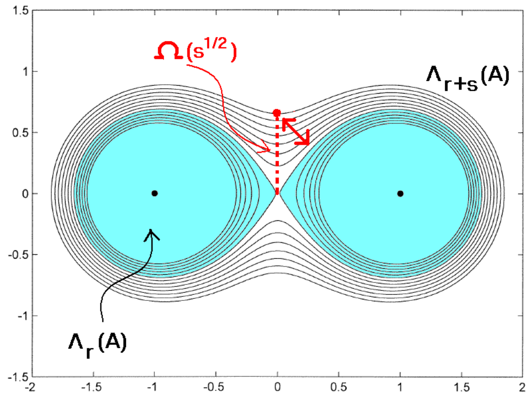

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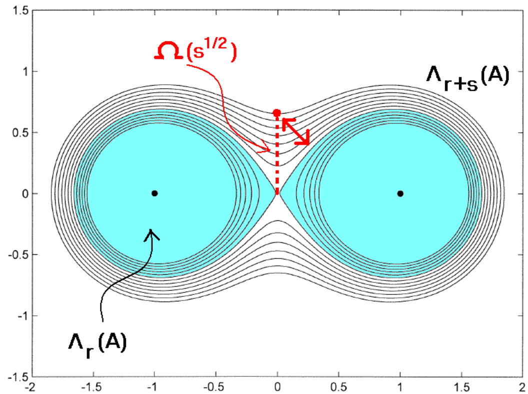

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Computing δ is tractable but impractical: $O(m^6)$ **(Gu 2000)**.

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(We'll prove a slightly weaker version. . .)

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Theorem (Milnor 1964)

If $p : \mathbf{R}^2 \to \mathbf{R}$ is a polynomial of degree d, then

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This **assumes** $\lambda_{\mathsf{min}}((P + zQ)(P + zQ)^*)$ **simple** $\forall z \in \mathbf{C}$.

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- Apply the "typical" result for (P_r, Q_r).
- Use **lower semicontinuity** of $\#(\cdot)$ on compact sets.

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- Optimizing pseudospectra is feasible computationally, and avoids these difficulties, by the Kreiss Matrix Theorem.
- The distance to uncontrollability can be computed polynomially by globally minimizing a bivariate function with simple level sets.