CONVEXITY AND EIGENVALUES OF SYMMETRIC MATRICES

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Technion: Lecture 1

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- Hyperbolic polynomials and the Lax conjecture

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- Some Lie algebra...

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True: Helton/Vinnikov 2002, Lewis/Parrilo/Ramana 2004.

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(Damped Newton's method for penalized version

$$\min\{\langle c, x\rangle - \mu \log p(x) : Ax = b\}, \qquad \text{as } \mu \downarrow 0.)$$

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Hence **semidefinite programming**:

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A powerful, tractable generalization of linear programming (eg: Ben-Tal/Nemirovski 2001).

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Since the Lax conjecture is true, all three-dimensional hyperbolicity cones are **semidefinite slices**:

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Are all hyperbolicity cones projections of semidefinite slices?

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This result extends to hyperbolic polynomials p (relative to d), interpreting $\{\lambda_i(x)\}$ as the roots of $t \mapsto p(x - td)$ (Bauschke/Güler/Lewis/Sendov 2001).

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Reminiscent of a famous result of von Neumann...

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Parallels von Neumann $\leftrightarrow$ Davis run deeper...

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(by a variational argument),

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$$G \;=\; (G|_{\mathbf{D}^n})_{**} \circ \mathrm{Diag} \circ \sigma \;=\; ((G|_{\mathbf{D}^n})_* \circ \mathrm{Diag} \circ \sigma)_*$$

(by a variational argument), so G is a norm. $\square$

9. DUALITY

Von Neumann's proof was duality-based.

If $\mathbf{E}$ is a Euclidean space and $G : \mathbf{E} \to \mathbf{R}_+$ satisfies

$$G(\alpha X) = |\alpha| G(X) \quad (\alpha \in \mathbf{R},\ X \in \mathbf{E})$$
$$\{X : G(X) \leq 1\} \text{ bounded,}$$

then the **dual function**

$$G_*(Y) \ = \ \sup\{\langle X, Y \rangle : G(X) \leq 1\}$$

is a norm. Furthermore, G is a norm $\Leftrightarrow$ $G = G_{**}$.

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Note also the **duality formula** $\quad G_*|_{\mathbf{D}^n} = (G|_{\mathbf{D}^n})_*.$

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The **Fenchel conjugate** of a function $F : \mathbf{E} \to (-\infty, +\infty]$,

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What is the unifying thread?

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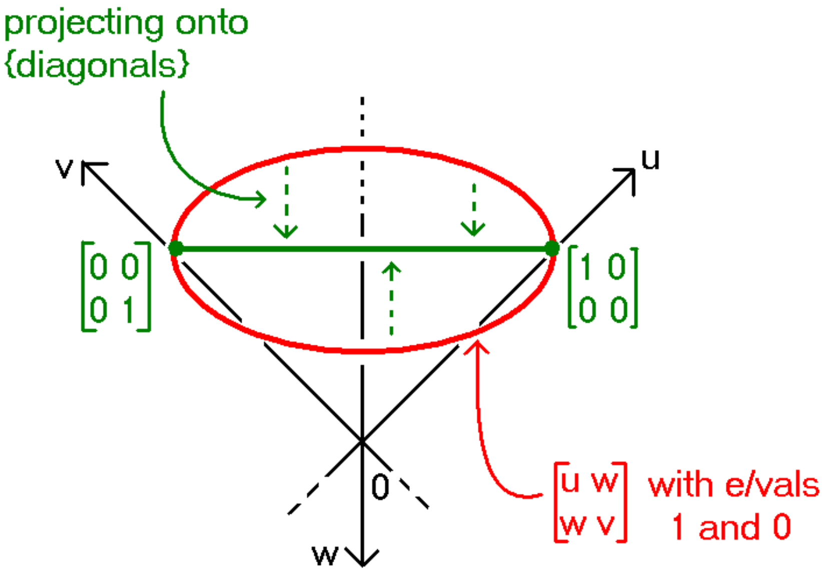

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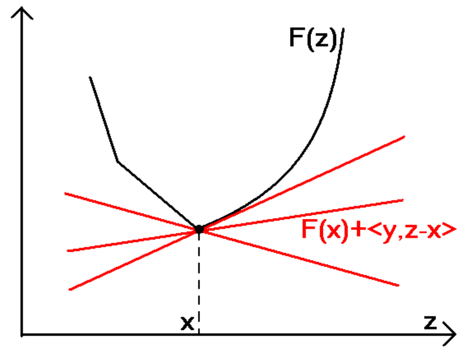

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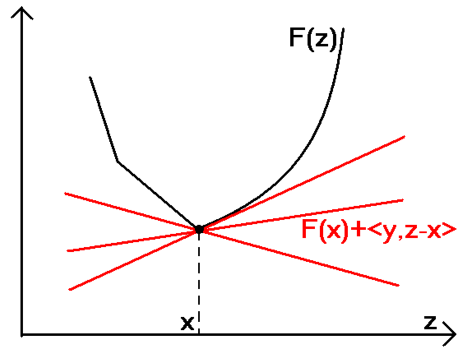

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(for Lipschitz f) etc...

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- Now apply the subgradient formula. $\square$

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