Eigenvalue optimization: symmetric versus nonsymmetric matrices

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- Robust stability and pseudospectra

2. EIGENVALUES AND MODELLING

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In optimization,

how do $\mathbf{M}_{-}^{n}$ and $\mathbf{S}_{-}^{n}$ compare?

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An even simpler question: **stable polynomials** $\mathbf{P}_{-}^{n}$ =

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$$z^n - \sum_{j=0}^{n-1} a_j z^j \ \in \ \mathbf{P}_-^n$$

exactly when

$$\begin{bmatrix} a_{n-1} & 1 & 0 & \cdots & 0 \\ a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \ddots & 1 \\ a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} \ \in \ \mathbf{M}_-^n.$$

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Conversely, $X \in \mathbf{M}^n_- \quad \Leftrightarrow \quad \det(zI - X) \in \mathbf{P}^n_-$.

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What about **local** optimization?

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For $A_1, \ldots, A_k, B \in \mathbf{S}^n$, suppose the semidefinite program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sum_j x_j A_j - B \;\in\; \mathbf{S}^n_- \end{array}$$

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Optimization with stable matrices (or polynomials) is similar.

For example...

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Problem: “Simultaneous plant stabilization” (Blondel, 1994).
Find (strictly) stable polynomials $p(z), q(z)$, and

$$(*) \qquad r(z) = (z^2 - 2\delta z + 1)p(z) + (z^2 - 1)q(z)$$

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An optimization approach:

$$\begin{array}{lrcl} \text{minimize} & \alpha & & \\ \text{subject to} & p, q & \in & \mathbf{P}^3_- \\ & r(z+\alpha) & \in & \mathbf{P}^5_- \\ & (*) \text{ holds.} & & \end{array}$$

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   $$\frac{dx}{dt} = \begin{bmatrix} -\epsilon & 1 & 0 \\ 0 & -\epsilon & 1 \\ 0 & 0 & -\epsilon \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 2e \end{bmatrix}.$$

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   Here, $x(t) \to 0$ **eventually**, but $x(\epsilon^{-1}) = \epsilon^{-2}$.

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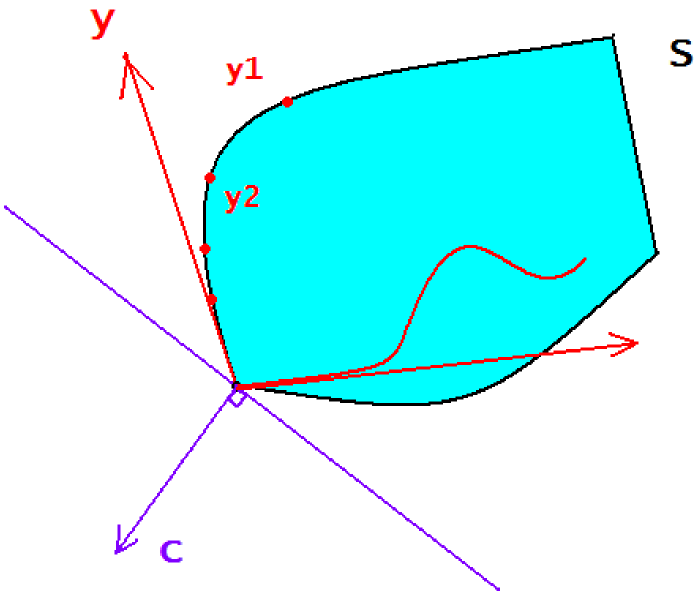
- find improving paths.

9. GEOMETRY OF TANGENTS



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Semi-algebraic sets: The g_i are polynomials (by the "curve selection lemma").

The stable sets $\mathbf{P}^n_-$ and $\mathbf{M}^n_-$ **are** semi-algebraic (by "quantifier elimination"), so tangents come from (analytic) paths.

11. NORMALS TO CONVEX SETS

Rewriting the optimality condition for maximizing $\langle c, \cdot \rangle$ over a closed convex set S gives a **normal cone** condition:

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Key property: the mapping $N_S(\cdot)$ is **closed** at every $x \in S$:

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Key property: the mapping $N_S(\cdot)$ is **closed** at every $x \in S$:

$$\begin{array}{rr} x_r \in S & x_r \to x \\ d_r \in N_S(x_r), & d_r \to d \end{array}$$

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How can we relate

- normal cone behavior for $\mathbf{S}^n_-$
- solution rank persistence in semidefinite programming?

12. NORMALS AND FACIAL STRUCTURE

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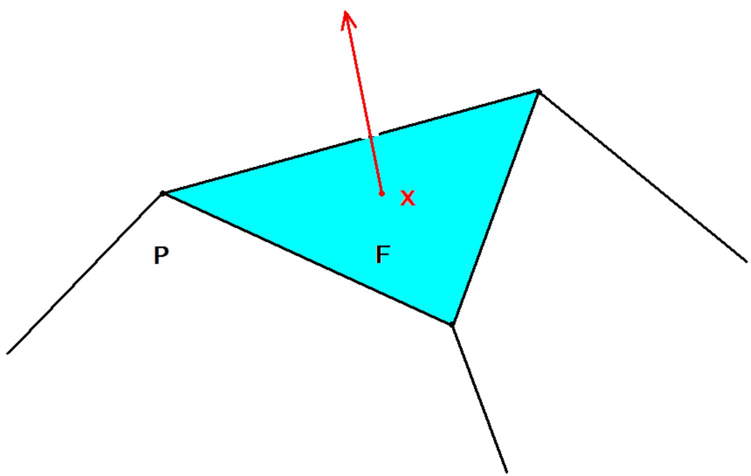
A polyhedron P decomposes into faces F of various dimensions. As $x \in F$ varies,

- $N_P(x)$ stays constant
- $N_P(x) - N_P(x) = N_F(x)$ (so P is "**sharp**" around F).



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So, $\mathcal{M}_r$ is an **identifiable surface** (Wright, 1993) of $\mathbf{S}^n_-$.

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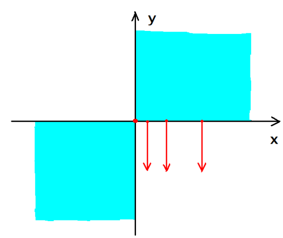
Convex sets are regular, but $\{(x,y) : xy \ge 0\}$ is **not** regular at the origin.



15. THE CHAIN RULE

Clarke regular sets have a nice calculus.

Theorem (Clarke, 1975) Suppose the map G is smooth and the set S is regular at $G(z)$, with

$$\mathrm{Null}(\nabla G(z)^*) \cap N_S(G(z)) \;=\; \{0\}.$$

Then the inverse image $G^{-1}(S)$ is regular at z, with

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Example A smoothly constrained set

$$F \;=\; \{x : g_i(x) \leq 0 \text{ for } i = 1, \ldots, m\}$$

is regular at x, with

$$N_F(x) \;=\; \left\{ \sum_i \lambda_i \nabla g_i(x) : \lambda \geq 0 \right\},$$

providing $\{\nabla g_i(x)\}$ linearly independent.

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Corollary The stable cone $\mathbf{M}^n_-$ is regular at every nonderogatory matrix (with normal cone...).

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The same chain rule holds for partly smooth sets.
So, $\mathbf{M}^n_-$ is also partly smooth, near nonderogatory matrices.

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How can we guarantee $x(c)$ is a local maximizer?

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Question Are $\mathbf{P}^n_-$ and $\mathbf{M}^n_-$ prox-regular?

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Aside: Checking stability of a "box" of polynomials is easy (Kharitonov, 1978).

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Quick to check via an algorithm of Boyd-Balakrishnan, 1990 .

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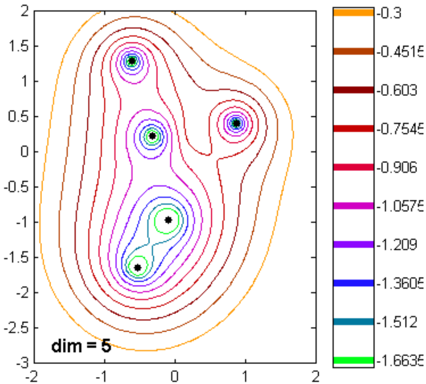

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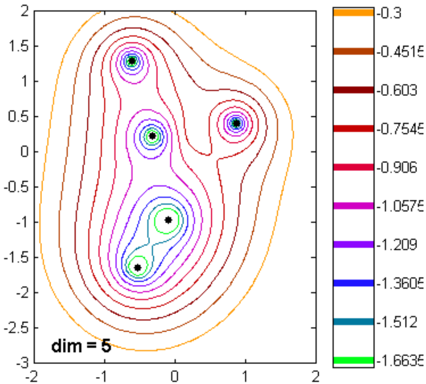

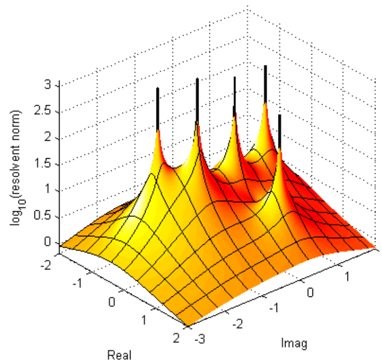

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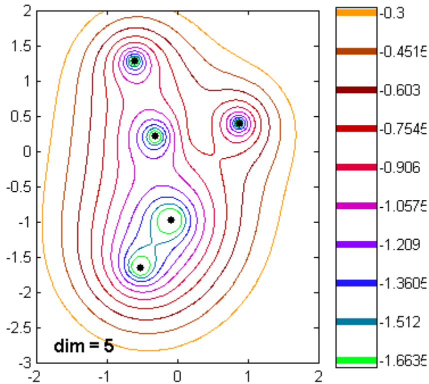

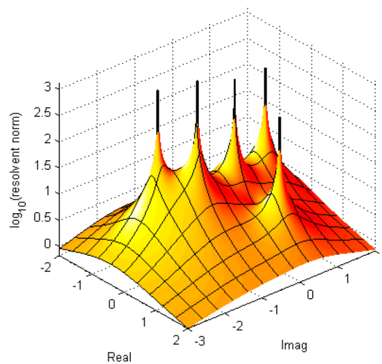

Demmel's example: $A = -\begin{bmatrix} 1 & 5 & 5^2 & 5^3 & 5^4 \\ 0 & 1 & 5 & 5^2 & 5^3 \\ 0 & 0 & 1 & 5 & 5^2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ldots$

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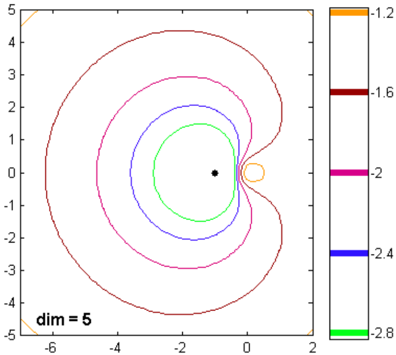

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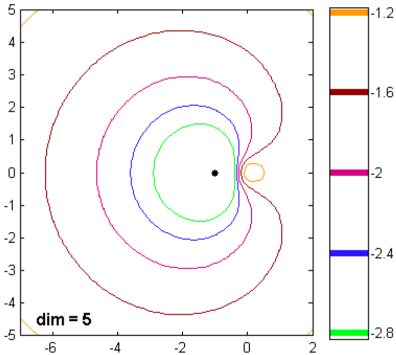

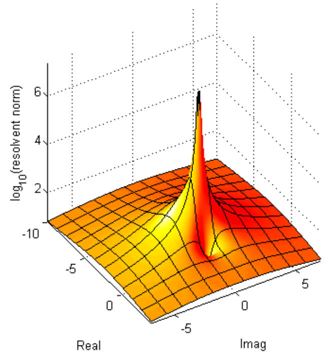

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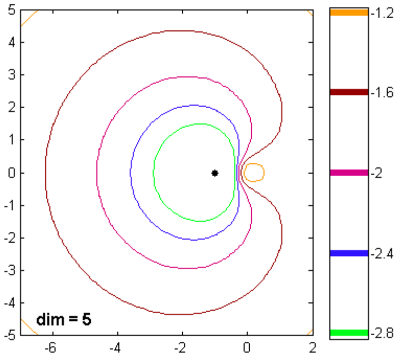

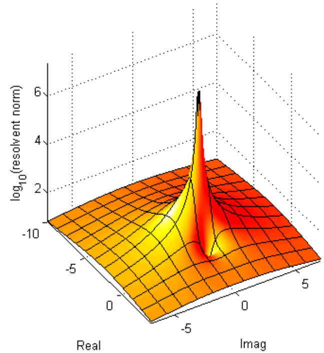

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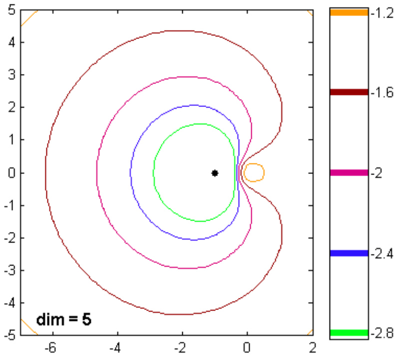

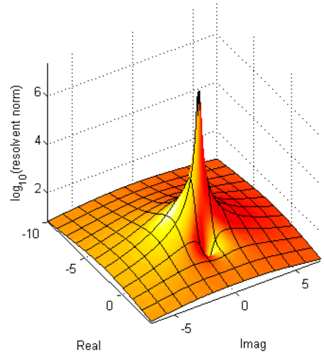

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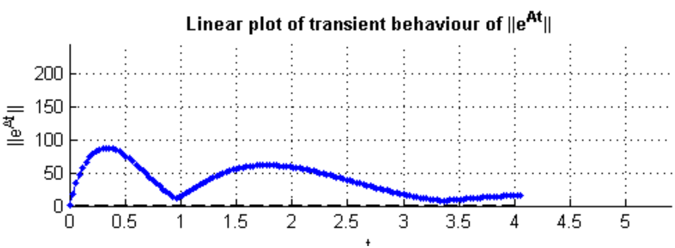

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We can compute α_e with a globally and locally quadratically convergent algorithm (like Boyd-Balakrishnan's), available in EigTool. We can even optimize it: Overton's talk.

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`orie.cornell.edu/∼aslewis`