EIGENVALUES AND OPTIMIZATION

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1. OUTLINE

PART I: SYMMETRIC MATRICES

- Hyperbolic polynomials, Lax, and convex optimization
- Convex spectral functions and unitarily invariant norms
- Duality, subgradients, and some Lie algebra

PART II: NONSYMMETRIC MATRICES

- Spectral radius, transient dynamics, and Kreiss's theorem
- Pseudospectral analysis and optimization
- Distance to uncontrollability

PART I:

SYMMETRIC MATRICES

2. HYPERBOLIC POLYNOMIALS

Example: Consider the homogeneous polynomial

$$p(u,v,w) = u^3 - 2uv^2 - uw^2 + 2v^2w.$$

For all real v, w, $\qquad p(u,v,w) = 0 \quad \Rightarrow \quad u$ **real**.

We say p is **hyperbolic relative to** $d = (1,0,0)$:
$t \mapsto p(x - td)$ always has all real roots.

Why?

$$p(u,v,w) = \det \begin{bmatrix} u & v & w \\ v & u & v \\ w & v & u \end{bmatrix}.$$

Lax Conjecture (1958) Hyperbolic polynomials on $\mathbf{R}^3$ relative to $(1,0,0)$ are all of the form

$$p(u,v,w) = \det(uI + vA + wB) \quad \text{with } A, B \text{ symmetric.}$$

True: Helton/Vinnikov 2002, Lewis/Parrilo/Ramana 2004.

3. HYPERBOLICITY CONES

Hyperbolic polynomials

- are simply defined;
- are common (there are open sets of such polynomials);
- have surprising convexity properties.

Theorem (Gårding 1951) Hyperbolic p relative to d has convex **hyperbolicity cone**—component of d in $\{x : p(x) > 0\}$.

Linear optimization over hyperbolicity cones is **tractable**!

Example: The **determinant** is hyperbolic relative to I on

$$\mathbf{S}^n \;=\; \{n \times n \text{ real symmetric matrices}\} :$$

each $X \in \mathbf{S}^n$ has all real eigenvalues $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$. The hyperbolicity cone is $\mathbf{S}^n_{++} = \{\text{positive definites}\}$.

Hence **semidefinite programming** (generalizing LP).

4. CONVEXITY AND SYMMETRY

Convexity of $\mathbf{S}^n_{++}$ and $-\log\det$ are special cases of:

Theorem (Davis 1957) Convexity and permutation-invariance of $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ $\Rightarrow$ convexity of

$$X \in \mathbf{S}^n \quad \mapsto \quad f(\lambda_1(X), \ldots, \lambda_n(X)).$$

(**Eg:** $f(x) =$

$$\begin{cases} 0 & (x > 0) \\ +\infty & (x \not> 0) \end{cases} \qquad \text{or} \qquad \begin{cases} -\sum_i \log x_i & (x > 0) \\ +\infty & (x \not> 0). \end{cases} \quad)$$

Extends to hyperbolic p (relative to d): interpret $\{\lambda_i(x)\}$ as the roots of $t \mapsto p(x - td)$ (Bauschke...2001). Similarly:

Lidskii's Theorem (extended) $\lambda(z) - \lambda(x)$ is a convex combination of permutations of $\lambda(z - x)$.

Lax conjecture reduces both results to $\mathbf{S}^n$ case (Gurvits 2004).

5. INVARIANCE

A function $F : \mathbf{S}^n \to \overline{\mathbf{R}}$ is **spectral** if

$$F(U^T X U) = F(X) \ \text{ whenever } \ U^T U = I$$

because then the spectral decomposition $\Rightarrow$

$$F(X) = F(\text{Diag}(\lambda_i(X))).$$

The Davis result **characterizes** convex spectral functions:

Theorem A spectral function F is convex $\Leftrightarrow$ F is convex on

$$\mathbf{D}^n = \{n \times n \text{ real diagonals}\}.$$

Reminiscent of...

Theorem (von Neumann 1937) Unitarily invariant
$G : \mathbf{M}^n \to \mathbf{R}$ is a norm $\quad\Leftrightarrow\quad$ the restriction $G|_{\mathbf{D}^n}$ is a norm.

6. DUALITY

Von Neumann's proof depended on a **duality formula**:

$$G_*|_{\mathbf{D}^n} = (G|_{\mathbf{D}^n})_*,$$

for the **dual function**

$$G_*(Y) \;=\; \sup\{\langle X, Y\rangle : G(X) \leq 1\}.$$

We can prove Davis's result by analogy. The **Fenchel conjugate** of a function $F : \mathbf{E} \to (-\infty, +\infty]$ is

$$F^*(Y) = \sup\{\langle X, Y\rangle - f(X)\},$$

and the duality formula

$$F^*|_{\mathbf{D}^n} = (F|_{\mathbf{D}^n})^*.$$

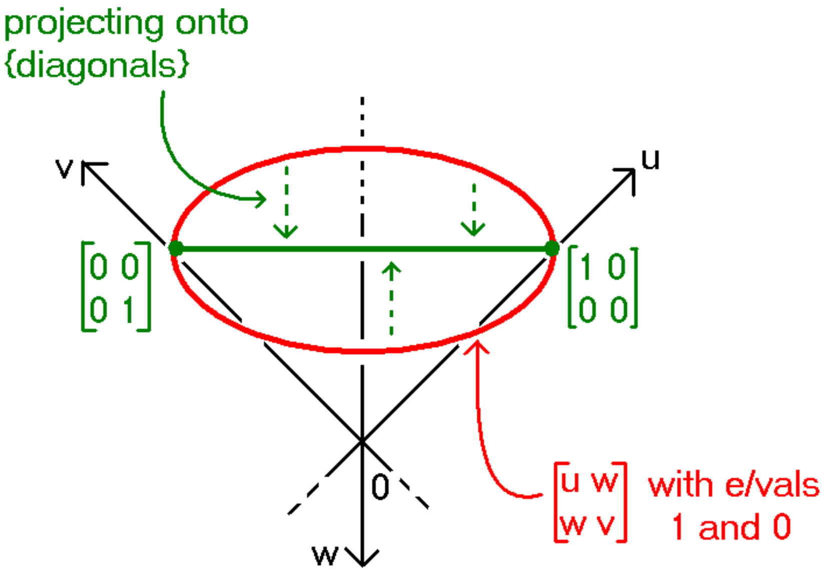
holds for spectral F.

What's the unifying thread?

7. HORN'S THEOREM (1954)

One route to the Davis result...

Theorem Convex combinations of permutations of $y \in \mathbf{R}^n$ give all possible diagonals of $X \in \mathbf{S}^n$ with eigenvalues $\{y_i\}$.



8. AN ALGEBRAIC FRAMEWORK

Theorem (Kostant 1973) Consider

- a real semisimple Lie group G, with Lie algebra $\mathfrak{g}$;
- a maximal compact subgroup $K \subset G$, with Lie algebra $\mathfrak{k}$;
- the corresponding **Cartan decomposition** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$;
- a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

Then for $x \in \mathfrak{a}$, $\quad \mathrm{proj}_{\mathfrak{a}}(K \cdot x) = \mathrm{conv}\,(W \cdot x)$,

where W is the **Weyl group**.

Example For Horn's theorem, take

$$\{U \in \mathbf{M}^n : U^T U = I,\ \det U = 1\} \quad \subset \quad \{U : \det U = 1\},$$

$$\{\text{skews}\} \oplus \{\text{traceless symmetric}\} \quad = \quad \{\text{traceless}\}$$

$$\mathfrak{a} = \{\text{diagonals}\} \ \text{ and } \ W = \{\text{permutations}\}.$$

9. CONVEXITY AND INVARIANCE

The von Neumann and Davis results extend to this setting:

Theorem Given a maximal compact subgroup K, and a maximal abelian subspace $\mathfrak{a}$ of a corresponding Cartan subspace $\mathfrak{p}$, consider an **invariant** function $F : \mathfrak{p} \to \overline{\mathbf{R}}$:

$$F(k \cdot x) = F(x) \quad (k \in K,\ x \in \mathfrak{p}).$$

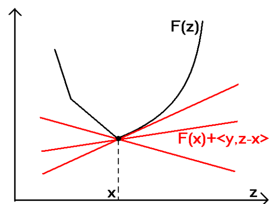
Then we have:

Convexity characterization F is convex $\Leftrightarrow$ $F|_{\mathfrak{a}}$ is convex.

Duality formula $F^*|_{\mathfrak{a}} = (F|_{\mathfrak{a}})^*$.

Convex subdifferential $y \in \partial F(x)$,

i.e. $F(x) + \langle y, z - x \rangle \leq F(z)\ \forall\ z \in \mathbf{E}$



$$\Leftrightarrow\ y = k \cdot v,\ x = k \cdot u,\ \text{with } k \in K,\ v \in \partial F|_{\mathfrak{a}}(u).$$

In the $\mathbf{S}^n$ case, this becomes...

10. SPECTRAL SUBDIFFERENTIALS

Theorem If $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is convex, permutation-invariant, so $X \in \mathbf{S}^n \mapsto F(X) = f(\lambda(X))$ is convex (by Davis), then: $Y \in \partial F(X) \quad \Leftrightarrow \quad$ **simultaneous spectral decomposition**,

$$U^T U = I, \quad U^T (\mathrm{Diag}\, x) U = X, \quad U^T (\mathrm{Diag}\, y) U = Y,$$

for some U and $y \in \partial f(x)$.

The subdifferential extends to nonconvex, Lipschitz f:

$$\partial f(x) \;=\; \mathrm{conv}\, \{ \lim \nabla f(x_r) : x_r \to x \}$$

(Clarke, 1974). The same characterization holds.

An illustration of nonsmooth analysis in linear algebra...

11. EIGENVALUE PERTURBATION THEORY

Theorem (Lidskii 1950) If $X, Z \in \mathbf{S}^n$, then $\lambda(Z) - \lambda(X)$ is a convex combination of permutations of $\lambda(Z - X)$.

A proof via nonsmooth analysis

- Via a separating hyperplane, we need, for any $w \in \mathbf{R}^n$

  $$w^T(\lambda(Z) - \lambda(X)) \;\le\; [w]^T \lambda(Z - X),$$

  where $w \mapsto [w]$ maps components into decreasing order.

- Consider the (nonconvex) spectral function

  $$F(X) = w^T \lambda(X).$$

  A **nonsmooth mean value theorem** shows

  $$F(Z) - F(X) = \langle Y, Z - X \rangle$$

  for some $Y \in \partial F(W)$ where $W \in [X, Z]$.

- Now apply the subdifferential formula. □

PART II:

NONSYMMETRIC MATRICES

12. THE SPECTRAL RADIUS

Question How should we design a (parametrized) square matrix A to force $A^n \to 0$ quickly as $n \to \infty$?

Theorem Rate of decay

$$\inf\{\mu : A^n = O(\mu^n) \text{ as } n \to \infty\}$$

equals **spectral radius**

$$\rho(A) \;=\; \max\{|\lambda| : \lambda \in \Lambda(A)\},$$

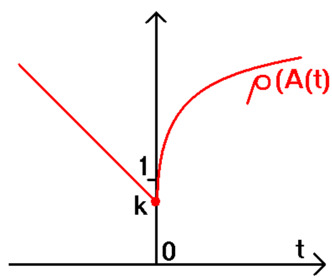
where $\Lambda(A) = \{\text{eigenvalues of } A\}$ is the **spectrum**.

Example

The spectral radius of

$$A(t) = \begin{bmatrix} k & 1 \\ t & k-t \end{bmatrix}$$

(with k slightly less than 1) is minimized at $t = 0$.

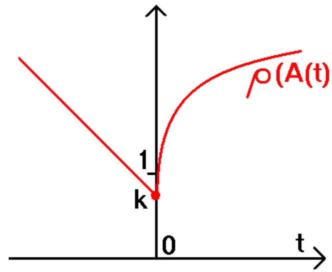


13. ROBUSTNESS AND TRANSIENT PEAKS

But

$$A(0) = \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}$$

may be unsatisfactory.



Difficulty I: $\rho(A(t))$ is highly sensitive to perturbation at $t = 0$ (nonlipschitz).

Difficulty II: The trajectory $\{A(0)^n\}$ has a big **transient peak**:

$$\begin{bmatrix} \frac{n}{n+1} & 1 \\ 0 & \frac{n}{n+1} \end{bmatrix}^n \sim e^{-1} \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix} \quad \text{for large } n.$$

One difficulty is the **multiple eigenvalue**. But this is **typical** at optimal solutions of spectral radius minimization problems. (Burke/Lewis/Overton 2001)

14. PSEUDOSPECTRA

(Following Trefethen. . .)

A powerful tool to visualize robust properties of eigenvalues:

$$\Lambda_\epsilon(A) \;=\; \bigcup_{\|X-A\|\le\epsilon} \Lambda(X) \;=\; \{z \in \mathbf{C} : \sigma_{\min}(A - zI) \le \epsilon\},$$

where $\sigma_{\min}$ is the **smallest singular value**.

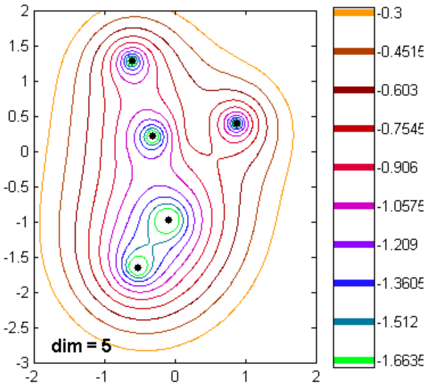
Pseudospectra resolve Difficulty II (transient peaks). . .

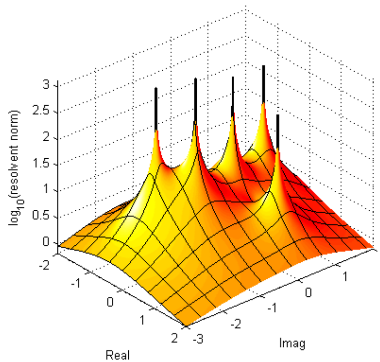
Kreiss Matrix Theorem (1962)
$A^n < K\rho^n$ for all n, with K not too large $\quad\Leftrightarrow$
$\max\{|\lambda| : \lambda \in \Lambda_\epsilon(A)\} \;<\; \rho$, with ϵ not too small.

Analogously, in **continuous time**, $e^{At} \to 0$ with peaks not too large when $\Lambda_\epsilon(A)$ lies in the left halfplane for ϵ not too small.

15. EXAMPLES

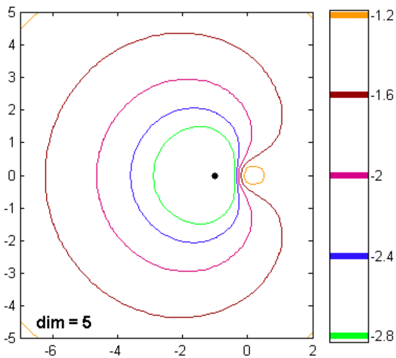
Pseudospectra for a random 5×5 triangular complex matrix, plotted by **T. Wright's EigTool**:

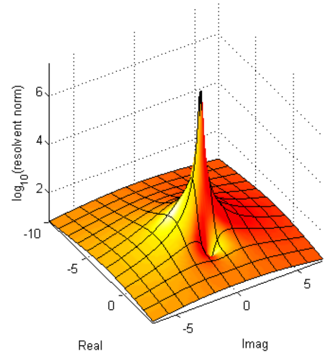




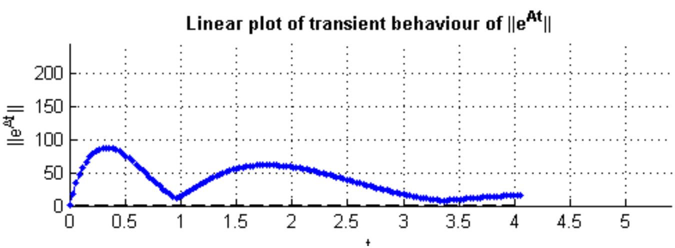
Demmel's example: $$A = -\begin{bmatrix} 1 & 5 & 5^2 & 5^3 & 5^4 \\ 0 & 1 & 5 & 5^2 & 5^3 \\ 0 & 0 & 1 & 5 & 5^2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ldots$$

16. A NEARLY UNSTABLE MATRIX





Notice $\Lambda_{.01}(A)$ extends outside the left halfplane: some "unstable" X satisfies $\|X - A\| \le .01$.

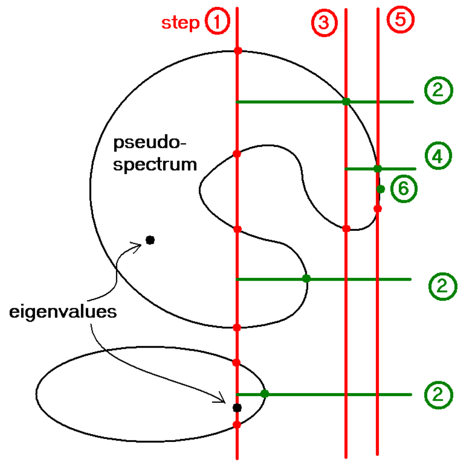


17. COMPUTING WITH PSEUDOSPECTRA

We can also **compute** pseudospectral quantities like

$$\alpha_\epsilon(A) \; = \; \max\{\mathrm{Re}\,\lambda : \lambda \in \Lambda_\epsilon(A)\}.$$

Key pseudospectral properties:



- Components all contain eigenvalues (by maximum modulus principle)
- Intersections with lines (or circles) easy to compute by a Hamiltonian eigensolver.

Hence a fast, accurate, robust **criss-cross algorithm** for α_ϵ: globally and quadratically convergent, available in `eigtool`.

Returns $\nabla \alpha_\epsilon$ (when it exists) $\rightarrow$ nonsmooth **gradient sampling** for **optimizing** α_ϵ (Burke/Lewis/Overton 2003).

18. LIPSCHITZ BEHAVIOR

Difficulty I? $A \mapsto \Lambda(A)$ isn't locally **Lipschitz**: no k satisfies

$$d(\Lambda(X), \Lambda(Y)) \;\leq\; k\|X - Y\| \quad \text{for all } X, Y \text{ near } A,$$

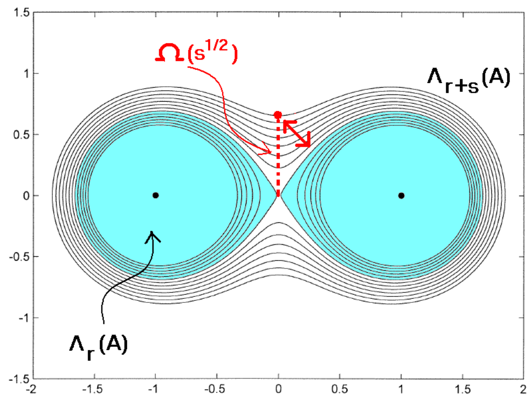
where d is Hausdorff distance between $U, V \subset \mathbf{C}$.

The pseudospectral map $A \mapsto \Lambda_\epsilon(A)$ can also be nonlipschitz:

$$A \;=\; \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$r \;=\; \frac{\sqrt{5}-1}{2}.$$

But. . .



Theorem Typically (Arnold 1971) A has all eigenspaces one-dimensional. Then, Λ_ϵ is Lipschitz around A for all small $\epsilon > 0$.

19. CONTROLLABILITY

A **system** with **state** $x \in \mathbf{C}^m$ and **control** u,

$$\frac{dx}{dt} = Ax + Bu,$$

controllable if any endpoints can be interpolated by a control.

Lemma (Hautus 1969) A matrix-pair (A, B) is controllable

$$\Leftrightarrow \quad \delta \;=\; \min\{\sigma_{\mathsf{min}}[A - zI, B] : z \in \mathbf{C}\} \;>\; 0.$$

Eising (1984) showed δ is the **distance to uncontrollability**:

$$\delta \;=\; \min\{\|(X, Y)\| : (A + X, B + Y) \text{ is uncontrollable}\}$$

Polynomially computable (Gu 2000, 2005) but tough...

Question (Trefethen) How many components has the **rectangular pseudospectrum**

$$\{z \in \mathbf{C} : \sigma_{\mathsf{min}}[A - zI, B] \le \epsilon\}?$$

20. CONNECTED COMPONENTS

Problem: For m-by-n P, Q, bound number of components

$$c \ = \ \#\{z : \sigma_{\mathrm{min}}(P + zQ) \le \epsilon\}.$$

Conjecture: $c \le m$. (Easy if $m = 1$ or $m = n$.)

Theorem $c \le 2m^2 - m + 1$. (Burke/Lewis/Overton 2004)

Proof Von Neumann-Wigner (1929) showed

$$\{m\text{-by-}m \text{ Hermitians with multiple eigenvalues}\}$$

has codimension 3. Assume (P, Q) "typical", so

$$\lambda_{\mathrm{min}}((P + zQ)(P + zQ)^*) \ \text{ simple } \ \forall z \in \mathbf{C}.$$

Milnor (1964) showed degree-d polynomials p on $\mathbf{R}^2$ satisfy

$$\#\{(x, y) : p(x, y) = 0\} \ \le \ d(2d - 1),$$

Result follows by applying this to $p : \mathbf{C} \cong \mathbf{R}^2 \to \mathbf{R}$ given by

$$p(z) \ = \ \det((P + zQ)(P + zQ)^* - \epsilon^2 I).$$

General case follows by perturbation.

21. SUMMARY

- Hyperbolicity is elegant for primal convex optimization.
  - But no apparent duality theory;
  - Is it really more general than semidefinite programming?
- Semisimple Lie theory gives a broad duality framework:
  - Fenchel conjugates;
  - convex and nonconvex subdifferentials.
- Spectral radius minimization leads to
  - multiple eigenvalues,
  - nonrobust solutions,
  - transient peaks.
- Pseudospectral optimization circumvents these (by Kreiss).
- Distance to uncontrollability is tractable.