The mathematics of eigenvalue optimization

Adrian Lewis

Simon Fraser University

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1. OUTLINE

Part I: Some history

Von Neumann and invariant matrix norms.

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Part IV: Stability of nonsymmetric matrices

Stable matrices; robust stability.

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PART I

SOME HISTORY

2. VON NEUMANN AND INVARIANT NORMS

$\mathbf{M}^n = \{n \times n \text{ matrices}\}$ unitaries $\mathbf{U}^n = \{U : U^*U = I\}$.
What norms on $\mathbf{M}^n$ are unitarily invariant:

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Eg: $\|X\|_2 = \|\sigma(X)\|_2$.

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via optimality conditions for

$$\max\{\langle Y, UXV\rangle : U, V \in \mathbf{U}^n\}.$$

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- optimization over matrix manifolds

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Theorem Suppose g a symmetric gauge. Then

$$Y \in \partial(g \circ \sigma)(X)$$

$$\Updownarrow$$

$$Y = U(\operatorname{Diag} y)V \quad \text{and} \quad X = U(\operatorname{Diag} x)V$$

$$\text{with } y \in \partial g(x), \ U, V \in \mathbf{U}^n.$$

(Watson 1992, Zietak 1993...)

PART II

SYMMETRIC MATRICES

3. CONVEX SPECTRAL FUNCTIONS

$\mathbf{S}^n = \{\text{symmetrics}\}$ orthogonals $\mathbf{O}^n = \{U : U^T U = I\}$.
What convex functions $F : \mathbf{S}^n \to \overline{\mathbf{R}}$ are spectral:

$$F(U^T X U) = F(X) \quad \forall\, U \in \mathbf{O}^n,\ X \in \mathbf{S}^n \ ?$$

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where $f(x) = F(\operatorname{Diag} x)$ symmetric on $\mathbf{R}^n$:

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Theorem (**Davis, 1957**) Convex spectral functions are exactly symmetric convex functions of eigenvalues.

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For tractability in optimization, consider semidefinite-representable (SDR) sets:

$$\{x \in \mathbf{R}^m : \mathcal{A}x \in B + L + \mathbf{S}^n_+\}$$

for linear $\mathcal{A} : \mathbf{R}^m \to \mathbf{S}^n$, subspace $L \subset \mathbf{S}^n$, $B \in \mathbf{S}^n$, and

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Eg: $$F(X) = \mathrm{tr}\,(X^{-1}) = \sum_i \frac{1}{\lambda_i(X)} \qquad (X \in \mathbf{S}^n_+).$$

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How can we unify?

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Eg: $\mathbf{E} = \mathbf{S}^n$, $p = \det$, $d = I$ gives

characteristic roots ↔ eigenvalues, $\qquad C_p = \mathbf{S}^n_+$.

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Hyperbolic polynomials have no obvious duality theory. So. . .

5. DUALITY AND LIE THEORY

One approach to convexity of spectral functions:

Theorem (**Horn, 1954**) If $x \in \mathbf{R}^n$,

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Theorem (**Kostant, 1973**) Consider maximal compact subgroup K of a real semisimple Lie group, giving Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. For $x \in \mathfrak{a}$,

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For Horn's case, take

$$\{U \in \mathbf{O}^n : \det U = 1\} \quad \subset \quad \{U \in \mathbf{M}^n : \det U = 1\},$$

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Theorem (**Horn, 1954**) If $x \in \mathbf{R}^n$,

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Theorem (**Kostant, 1973**) Consider maximal compact subgroup K of a real semisimple Lie group, giving Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. For $x \in \mathfrak{a}$,

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$$\mathfrak{a} = \{\text{diagonals}\} \quad \text{and} \quad W = \mathbf{P}^n.$$

INVARIANT CONVEX FUNCTIONS

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Restating von Neumann and Davis:

$$\begin{aligned} \text{unitarily invariant } F \text{ a norm on } \mathbf{M}^n &\Leftrightarrow F|_{\mathbf{D}^n} \text{ a norm} \\ \text{spectral } F \text{ convex on } \mathbf{S}^n &\Leftrightarrow F|_{\mathbf{D}^n} \text{ convex} \end{aligned}$$

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Extends?

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and if F also convex we have subdifferential formula

$$y \in \partial F(x)$$

$$\Updownarrow$$

$$y = k \cdot v \quad \text{and} \quad x = k \cdot u$$

$$\text{with } v \in \partial F|_{\mathfrak{a}}(u), \;\; k \in K.$$

(Subsumes case of unitarily invariant norms.)

PART III

SOME VARIATIONAL ANALYSIS

6. NONSMOOTH OPTIMIZATION

(Clarke, Ioffe, Mordukhovich, Rockafellar. . .)

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For nonconvex $f : \mathbf{E} \to \overline{\mathbf{R}}$, $y \in \hat{\partial} f(x)$ means $f(x)$ finite and

$$\langle y, z - x\rangle \le f(z) - f(x) + o(\|z - x\|) \quad \text{as } z \to x.$$

(Unifies convex subdifferential and Fréchet derivative.)

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To "stabilize" this idea, define $y \in \partial f(x)$ if

$$\exists\ x_r \to x,\ y_r \to y,\ \text{so } f(x_r) \to f(x),\ y_r \in \hat{\partial} f(x_r).$$

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Convex sets and smooth manifolds are regular, with obvious normal cones.

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Mean value theorem If $f : \mathbf{E} \to \mathbf{R}$ Lipschitz around a line $[x, y]$, then

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Chain rule For smooth $\Phi : \mathbf{R}^m \to \mathbf{E}$, if $S \subset \mathbf{E}$ regular at $\Phi(v)$ and

$$N_S(\Phi(v)) \cap \ker(\nabla\Phi(v))^* = \{0\},$$

then $\Phi^{-1}(S)$ regular at v, and

$$N_{\Phi^{-1}(S)}(v) = (\nabla\Phi(v))^* N_S(\Phi(v)).$$

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PARTLY SMOOTH SETS

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Sensitivity analysis needs smooth *and* nonsmooth analysis. . .

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7. EIGENVALUE PERTURBATION THEORY

A new perspective, via nonsmooth analysis.

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where $w \mapsto \bar{w}$ orders components decreasingly.

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using nonconvex version of subdifferential formula

$$Y \in \partial(f \circ \lambda)(X)$$
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$$Y = U^*(\operatorname{Diag} y)U \;\text{ and }\; X = U^*(\operatorname{Diag} x)U$$
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8. STABLE POLYNOMIALS

In

$$\mathcal{P}^n = \{\text{complex polynomials, degree } \leq n\},$$

consider the stable monics

$$\mathcal{P}^n_{\text{st}} = \big\{p \in \mathcal{P}^n : z^{n+1} + p(z) = 0 \;\Rightarrow\; \mathrm{Re}\, z \leq 0\big\}.$$

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Theorem $\mathcal{P}^n_{\text{st}}$ is everywhere regular (with normal cone. . .)

(Burke/Overton, 2001)

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In fact, $\mathcal{P}^n_{\text{st}}$ is partly smooth relative to manifolds of p having imaginary zeros of $z^{n+1} + p(z)$ of fixed multiplicity.

So, these multiplicities persist at solutions of optimization problems over $\mathcal{P}^n_{\text{st}}$.

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Conformally map left halfplane to unit disk by

$$w \mapsto \frac{w+1}{w-1} \qquad (w \in \mathbf{C}).$$

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Since Φ smooth with $\nabla\Phi$ onto, chain rule shows

$$\mathcal{P}^n_{\mathrm{pst}} = \Phi^{-1}(\mathcal{P}^n_{\mathrm{st}})$$

everywhere regular, with normal cone...

PART IV

STABILITY OF NONSYMMETRIC MATRICES

With Jim Burke and Michael Overton

9. STABLE MATRICES

Fundamental for dynamical systems are the stable matrices:

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Trajectories of $\frac{dx}{dt} = Ax$ grow exponentially $\quad\Leftrightarrow\quad A \notin \mathbf{M}^n_{\mathsf{st}}$.

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Unlike in $\mathbf{S}^n$, **typical** eigenvalues λ of $A \in \mathbf{M}^n$ (even if algebraically multiple) are nonderogatory:

$$\dim\ker(\lambda I - A) = 1. \qquad \textbf{(Arnold, 1971)}$$

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Similar argument works for power-stable matrices (eigenvalues all satisfy $|\lambda| \leq 1$).

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- $\exists H \succ 0$ so $H \succ A^*HA$

(where $H \succ G$ for Hermitians means $H - G$ positive definite).

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- non-robust eigenvalues near unit circle.

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Useful for viewing pseudospectra is T. Wright's EigTool.

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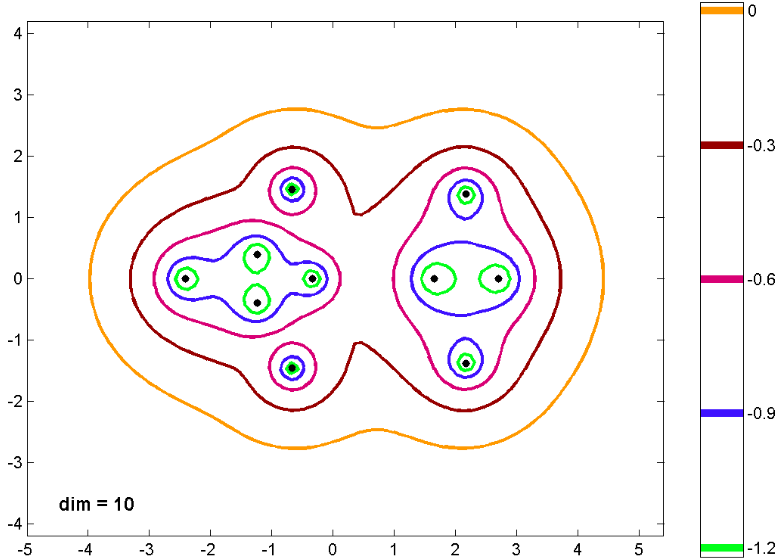
PSEUDOSPECTRAL PLOT

Pseudospectra for a random upper-triangular 10-by-10 real matrix:



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Hence: pseudopsectral behaviour measures transient peaks.

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Continuous-time analogue: the pseudospectral abscissa

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- distance to instability, for **some** ϵ.

11. OVERVIEW

Eigenvalue optimization blends diverse mathematical threads.

- Elegant classical mathematics:
  - matrix analysis;
  - eigenvalue perturbation theory.
- Important and challenging applications (like robust stability and control).
- A natural testing ground for modern nonsmooth optimization:
  - theory;
  - computational practice.

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