A nonsmooth Morse-Sard theorem for subanalytic functions

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Abstract According to the Morse-Sard theorem, any sufficiently smooth function on a Euclidean space remains constant along any arc of critical points. We prove here a theorem of Morse-Sard type suitable as a tool in variational analysis: we broaden the definition of a critical point to the standard notion in nonsmooth optimization, while we restrict the functions under consideration to be semi-algebraic or subanalytic. We make no assumption of subdifferential regularity. Łojasiewicz-type inequalities for nonsmooth functions follow quickly from tools of the kind we develop, leading to convergence theory for subgradient dynamical systems.

Key words Critical point, nonsmooth analysis, non-regular function, Morse-Sard theorem, semialgebraic function, subanalytic function.

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1 Introduction

Variational analysts broaden the classical notion of a critical point of a smooth function on a Euclidean space to deal with the kinds of lower semicontinuous functions typical in nonsmooth optimization. Standard nonsmooth theory replaces the gradient of a smooth function f at a point x with a set known as the "(limiting) subdifferential", $\partial f(x)$; if $0 \in \partial f(x)$, we call x "(lower) critical". Our aim in this work is to develop a version of the famous Morse-Sard theorem suitable as a tool in this nonsmooth context.

The subdifferential is not as immediately intuitive as the gradient. As a simple example, consider the (semi-algebraic) function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = 2\max\left\{x_1, \min\{x_2, x_3\}\right\} - x_2 - x_3 \qquad (1)$$

The subdifferential $\partial f(0)$ is the union of the two line segments $[2e_1 - e_2 - e_3, e_2 - e_3]$ and $[2e_1 - e_2 - e_3, e_3 - e_2]$ (where e_i is the ith unit vector), so 0 is not a lower critical point. On the other hand, $\partial(-f)(0) = \{e_2 + e_3 - 2e_1\} \cup [e_2 - e_3, e_3 - e_2]$, so 0 *is* a lower critical point of $-f$. Despite its challenges, however, the subdifferential has proved a powerful foundation for nonsmooth optimization and control theory [5, 18].

Our goal of understanding Morse-Sard-type results in a nonsmooth setting is driven in part by the broad success of the subdifferential as an analytic tool. In part, the results we develop here also support specific applications for nonsmooth Łojasiewicz-type inequalities, leading to convergence theory for subgradient dynamical systems [4].

We begin by recalling the classical Morse-Sard theorem. The set of *critical points* of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, denoted by $\operatorname{crit} f$, is the subset of $\mathbb{R}^n$ on which all first order partial derivatives of f vanish. Its image $f(\operatorname{crit} f)$ is called the set of *critical values*. With this terminology, a k-time continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ – denoted $f \in C^k(\mathbb{R}^n)$ – is said to have the *Sard property* if the set of its critical values has zero Lebesgue measure. The Morse-Sard theorem ([15], [19]) asserts in particular that every $C^m(\mathbb{R}^n)$ function, $m \geq n$, has the Sard property.

The celebrated example of Whitney [21] of a smooth function not constant on an arc of its critical points reveals a typical failure of the Sard property. This failure might occur when the following two conditions are met: the function has a low order of smoothness (that is, strictly less than the dimension of the space) and the set of critical points is "pathological", see Hajlasz [8].

In order to circumvent the strong smoothness properties required by the classical Morse-Sard theorem, various other conditions can supplant the double smoothness/dimension assumption recalled above. Existing deep results restrict attention to particular subclasses of functions (semi-algebraic or "o-minimal", for example) [3, 6, 10], distance functions to a manifold [17], or nonsmooth functions admitting a supremum representation [22], in order to relax the smoothness condition into simple differentiability or even into some kind of tractable non-differentiability hypothesis.

Our interest is in extended-real-valued continuous subanalytic functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. In Subsection 2.1 we recall general facts in subanalytic geometry. The nonsmooth aspects and the occurrence of infinite values require the choice of a notion of critical point. As explained above, in the present note, we work with the limiting subdifferential (Definition 6(ii)), see Mordukhovich [13]. We recall the definition of a (lower) critical point as well as basic nonsmooth calculus rules in Subsection 2.2.

If we are prepared to assume that the nonsmooth subanalytic function f is "subdifferentially regular" (see Subsection 2.2 for the definition), a simple application of a standard nonsmooth chain rule shows that f is constant on the set of its lower critical points (Corollary 11). Subdifferential regularity, however, is a strong assumption: it fails for functions as simple as $-\|\cdot\|$.

Our main results (Theorem 13 and Theorem 14) dispense with any assumption of subdifferential regularity, relying only on continuity. In this case, nonsmooth chain rules appear unhelpful. Not surprisingly, our proof does rely on the standard fundamental structural result about subanalytic functions, which "stratifies" the graphs of such functions into smooth manifolds. However, example (1) illustrates the challenge in proving such results: even given an (obvious) stratification, the behavior of the subdifferential may not be transparent. Our proof also relies on Pawłucki's generalization of the Puiseux lemma [16, Proposition 6].

One of our main motivations for establishing a result of Morse-Sard type (like Theorem 14) for subanalytic continuous functions is their relationship with the generalized Łojasiewicz inequality for continuous subanalytic functions established in [4, Theorem 3.5]. Specifically, in Theorem 15, we observe:

> A continuous subanalytic function satisfies the generalized Łojasiewicz inequality if and only if it has the Sard property.

2 Preliminaries

In Subsection 2.1 we recall basic properties of subanalytic sets and functions, which can be found for instance in Bierstone-Milman [2], Łojasiewicz [12] or Shiota [20]. For the particular case of semialgebraic functions, we refer to the textbooks of Benedetti-Risler [1] and Bochnak-Coste-Roy [3]. For the more general framework of o-minimal structures, see Dries-Miller [7] or Coste [6].

Subsection 2.2 contains some prerequisites from variational and nonsmooth analysis. These can be found for example in the books of Clarke-Ledyaev-Stern-Wolenski [5] or Rockafellar-Wets [18].

2.1 Elements from real subanalytic geometry

Let us recall some basic notions.

Definition 1 (subanalyticity) (i). A subset A of $\mathbb{R}^n$ is called *semianalytic* if each point of $\mathbb{R}^n$ admits a neighborhood V for which $A \cap V$ assumes the following form

$$\bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in V : f_{ij}(x) = 0, g_{ij}(x) > 0\},$$

where the functions $f_{ij}, g_{ij} : V \mapsto \mathbb{R}$ are real-analytic for all $1 \le i \le p, 1 \le j \le q$.
(ii). The set A is called *subanalytic* if each point of $\mathbb{R}^n$ admits a neighborhood V such that

$$A \cap V = \{x \in \mathbb{R}^n : (x, y) \in B\}$$

where B is a bounded semianalytic subset of $\mathbb{R}^n \times \mathbb{R}^m$ for some $m \ge 1$.

(iii). Given two integers $m, n \geq 1$, a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (respectively, a point-to-set operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$) is called *subanalytic*, if its graph

$$\operatorname{Gr} f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) = \lambda\} \text{ (respectively, } \operatorname{Gr} T := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in T(x)\})$$

is a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}$ (respectively, of $\mathbb{R}^n \times \mathbb{R}^m$).

If a subset A of $\mathbb{R}^n$ is subanalytic then so are its closure $\operatorname{cl} A$, its interior $\operatorname{int} A$, and its boundary $\operatorname{bd} A$. Subanalytic sets are closed under locally finite union and intersection and the complement of a subanalytic set is subanalytic (the Gabrielov Theorem).

The image and the preimage of a subanalytic set are not in general subanalytic sets. This is essentially due to the fact that the image of an unbounded subanalytic set by a linear projection may fail to be subanalytic. Consider for instance the set $\{(n^{-1}, n) : n \in \mathbb{N}^*\}$, whose projection onto $\mathbb{R} \times \{0\}$ is not subanalytic at 0. Let us introduce a stronger analytic-like notion called "global" subanalyticity. For each $n \in \mathbb{N}$, set $C_n = (-1, 1)^n$ and define τ_n by

$$\tau_n(x_1, \ldots, x_n) = \left(\frac{x_1}{1 + x_1^2}, \ldots, \frac{x_n}{1 + x_n^2} \right).$$

Definition 2 (global subanalyticity) (e.g. [7, p. 506]) (i). A subset S of $\mathbb{R}^n$ is called *globally subanalytic* if $\tau_n(S)$ is a subanalytic subset of $\mathbb{R}^n$.
(ii). An extended-real-valued function (respectively, a multivalued mapping) is called *globally subanalytic* if its graph is globally subanalytic.

Let us recall briefly several classical facts concerning globally subanalytic objects.

- Globally subanalytic sets are subanalytic.

- Any bounded subanalytic set is globally subanalytic. Analytic functions are always subanalytic ([7, Fact 1.1]), but they might fail to be globally subanalytic (think of the graph of the sinus function) unless they are restricted to a bounded set.

- **(projection theorem)** ([7, Example 4, p. 505]) Let $S \subset \mathbb{R}^{n+1}$ be a globally subanalytic set and let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the canonical projection defined as usual by $\Pi(x_1 \ldots, x_{n+1}) = (x_1, \ldots, x_n)$. Then the projection of S onto $\mathbb{R}^n$, namely $\Pi(S)$, is a globally subanalytic subset of $\mathbb{R}^n$.

- The image and the preimage of a globally subanalytic set by a globally subanalytic function (respectively, globally subanalytic multivalued operator) is globally subanalytic (e.g. [7, p. 504]).

Semialgebraic sets and functions provide an important subclass of globally subanalytic objects. Recall that a set $A \subset \mathbb{R}^n$ is called *semialgebraic* if it assumes the following form

$$A = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in V : f_{ij}(x) = 0, g_{ij}(x) > 0\},$$

where $f_{ij}, g_{ij} : \mathbb{R}^n \mapsto \mathbb{R}$ are polynomial functions for all $1 \leq i \leq p, 1 \leq j \leq q$. As before, a function f is called *semialgebraic* if its graph is a semialgebraic set.

The Tarski-Seidenberg theorem (see [3] for instance) asserts that the class of semialgebraic sets is stable under linear projection.

Let us finally mention the following fundamental results that will be used in the sequel. The first one reflects properties of the well-behaved topology of subanalytic sets.

Proposition 3 (path connectedness) (e.g. [7, Facts 1.10-1.12]) *Any globally subanalytic* (*respectively, subanalytic*) *set has a finite* (*respectively, a locally finite*) *number of connected components. Moreover, each component is subanalytic and subanalytically path connected, that is, every two points can be joined by a continuous subanalytic path that lies entirely in the set.*

Subanalytic sets have a "good" structure. The meaning of "good" is made clear by the following proposition.

Proposition 4 (stratification) (e.g. [7, Fact 1.19]) *Let S be a globally subanalytic subset of $\mathbb{R}^n$ and let M and F be two subanalytic subsets of S. Then there exists a subanalytic stratification $\mathcal{P} = \{C_i\}_{i=1}^p$ of S compatible with M and F, that is:*

(i) each statum C_i is a subanalytic C^∞-manifold of dimension $0 \leq d_i \leq n$

(ii) $\bigcup_i C_i = S$

(iii) for each $i \neq j$ we have $C_i \cap C_j = \emptyset$ and

$$C_i \cap \mathrm{cl}C_j \neq \emptyset \implies C_i \subset \mathrm{cl}C_j \setminus C_j.$$

(iv) $\mathcal{P}$ is compatible with $\{M, F\}$, that is every stratum C_i is either included in F (resp. M) or has an empty intersection with F (resp. M).

Let us finally state the following remarkable property of one-variable continuous subanalytic functions.

Lemma 5 *Let $h : [0,1] \to \mathbb{R}$ be a continuous subanalytic function. Then h is absolutely continuous and differentiable (in fact, analytic) in a complement of a finite set.*

Proof. The function h is readily seen to be globally subanalytic, and the result follows from the monotonicity lemma (e.g. [7, Theorem 4.1], [6, Theorem 2.1]). $\diamond$

2.2 Elements from variational analysis

Let us recall several definitions and facts from variational and nonsmooth analysis.

Definition 6 (subdifferential) (e.g. [18, Definition 8.3])

(i). The Fréchet subdifferential $\hat{\partial} f(x)$ of a lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathrm{dom}\, f := \{x \in \mathbb{R}^n : f(x) \in \mathbb{R}\}$ is defined as follows:

$$\hat{\partial} f(x) = \left\{ x^* \in \mathbb{R}^n : \liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

When $x \notin \mathrm{dom}\, f$, we set $\hat{\partial} f(x) = \emptyset$.

(ii). The limiting subdifferential of f at $x \in \mathbb{R}^n$, denoted by $\partial f(x)$, is the set of all cluster points of sequences $\{x_n^*\}_{n \geq 1}$ such that $x_n^* \in \hat{\partial} f(x_n)$ and $(x_n, f(x_n)) \to (x, f(x))$ as $n \to +\infty$.

It is a well known result of variational analysis that $\hat{\partial} f(x)$ (and a fortiori $\partial f(x)$) is not empty in a dense subset of the domain of f (see [18], for example).

Remark 1 If an extended-real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has a closed domain $\mathrm{dom}\, f$ relative to which it is continuous (that is, $f\,|_{\mathrm{dom}\, f}$ is continuous), then f is lower semicontinuous and the graph of the limiting subdifferential ∂f is simply the closure of the graph of the Fréchet subdifferential $\hat{\partial} f$, that is,

$$\{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : x^* \in \partial f(x)\} = \mathrm{cl}\, \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : x^* \in \hat{\partial} f(x)\}. \quad (2)$$

The Fréchet and the limiting subdifferentials essentially reflect local variations of f from the viewpoint of its epigraph (see [18, Chapters 6-8]). Therefore a condition like $\partial f(a) \ni 0$ should rather be thought as a definition for a to be "lower critical". For instance the continuous function $N : x \mapsto -||x||$ admits 0 as a maximizer, whereas $\partial N(0) = S^{n-1}$ and thus 0 is not a lower-critical point. With this in mind let us give the following definition.

Definition 7 (lower critical point) A point $a \in \mathbb{R}^n$ is called a *lower critical point* of the lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ if $0 \in \partial f(a)$. In this case we denote $a \in L\text{-}\operatorname{crit} f$.

Remark 2 If $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, one can define similarly the notion of *upper* critical points. A point $a \in \mathbb{R}^n$ is called an *upper critical point* of f if $0 \in -\partial[-f](x)$, which we denote by $a \in U\text{-}\operatorname{crit} f$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, a point $a \in \mathbb{R}^n$ is called a (generalized) *critical point* of f if it belongs to the set[1]

$$\operatorname{crit} f = \{x \in \mathbb{R}^n : 0 \in \partial f(x) \cup [-\partial(-f)(x)]\}. \tag{3}$$

Remark 3 When f is finite-valued and C^1 one has

$$\partial f(x) = -\partial(-f)(x) = \{\nabla f(x)\},$$

so the notions of critical points introduced above all coincide with the usual one:

$$\operatorname{crit} f = \{x \in \mathbb{R}^n : \nabla f(x) = 0\}.$$

We next discuss the rather strong condition of subdifferential regularity.

Definition 8 (subdifferential regularity) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Define $\delta : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ by $\delta(u) = 0$ if $u \in \operatorname{epi} f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq \lambda\}$, and $\delta(u) = +\infty$ otherwise. The function f is called *subdifferentially regular at* $x \in \operatorname{dom} f$ if $\hat{\partial}\delta(x, f(x)) = \partial\delta(x, f(x))$, and *subdifferentially regular*, if it is subdifferentially regular throughout its domain.

Remark 4 Note that if f is subdifferentially regular then $\hat{\partial} f = \partial f$.

Let us recall the following easy property (see [18, Theorem 10.6, page 427], for example).

Proposition 9 (Chain rule) *Suppose that $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is such that $h(x) = f(g(x))$ where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and $g : \mathbb{R}^m \to \mathbb{R}^n$ is a C^1 function. Then for every point $x \in \operatorname{dom} h$ one has*

$$\nabla g(x)^T \hat{\partial} f(g(x)) \subset \hat{\partial} h(x),$$

where $\nabla g(x)^T$ denotes the transpose of the Jacobian matrix of g at x.

As a consequence of the projection theorem, we have the following stability results:

Proposition 10 *Let f be an extended-real-valued function.*

(i) If f is globally subanalytic, then the operators $\hat{\partial} f$ and ∂f and the set $L\text{-}\operatorname{crit} f$ are globally subanalytic.

(ii) If f is subanalytic and relatively bounded on its domain (that is, $\{f(x) : x \in \operatorname{dom} f \cap B\}$ is bounded for every bounded subset B of $\mathbb{R}^n$), then the operators $\hat{\partial} f$, ∂f and the set $L\text{-}\operatorname{crit} f$ are subanalytic.

Proof. See [4, Proposition 2.13]. ◇

Remark 5 If in addition the function f is continuous and finite-valued, then the same result holds for the sets $U\text{-}\operatorname{crit} f$ and $\operatorname{crit} f$.

The following result is an easy consequence of Proposition 9.

[1] The interested reader may compare with the notion of "symmetric subdifferential" introduced in [14, page 1265], and used to derive an exact mean-value theorem for a class of continuous functions, see [14, Theorem 7.9].

Corollary 11 *Assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a continuous subanalytic function which is subdifferentially regular. Then:*

(i) f is constant on every connected component of L-$\operatorname{crit} f$.

(ii) $f(L\text{-}\operatorname{crit} f)$ is countable, and thus of measure zero.

Proof. By Proposition 10 the set L-$\operatorname{crit} f$ is subanalytic, thus in view of Proposition 3 it has a countable number of connected components, which are subanalytically path connected. Thus (ii) is a direct consequence of (i). So, let us prove that f is constant on some connected component S of L-$\operatorname{crit} f$. To this end, let x, y be in S and consider a continuous subanalytic path $z : [0,1] \to S$ with $z(0) = x$ and $z(1) = y$. Define $h : [0,1] \to \mathbb{R}$ by $h(t) = (f \circ z)(t)$. Since f is subdifferentially regular we have $0 \in \partial f(z(t)) = \hat{\partial} f(z(t))$ for all $t \in [0,1]$. Applying Proposition 9 we get $0 \in \hat{\partial} h(t)$, for all but finitely many $t \in [0,1]$. Now by Lemma 5 and Remark 3 it follows that $\hat{\partial} h(t) = \{\dot{h}(t)\}$ for all t in the complement of a finite set, where $\dot{h}(t)$ denotes the derivative of h at the point t. It follows that h is constant and $f(x) = f(y)$. ◇

In the next section we will see that the conclusion of Corollary 11 is much more general and that the assumption "f is subdifferentially regular" is superfluous.

3 Main results

The following lemma is crucial for our considerations. It also has an independent interest.

Lemma 12 *Let F be a nonempty globally subanalytic subset of $\mathbb{R}^n$, $\gamma : [0,1] \to \mathrm{cl}F$ a one-to-one continuous subanalytic path and $\delta > 0$. Then there exists a continuous subanalytic path $z : [0,1] \to \mathrm{cl}F$ such that*

(i) $\int_0^1 \|\dot{z}(t) - \dot{\gamma}(t)\|\, dt < \delta$ (in fact, $\|\dot{z}(t) - \dot{\gamma}(t)\| < \delta$ for all but finitely many $t \in [0,1]$) ;

(ii) the (subanalytic) set

$$\Delta := \{t \in [0,1] : z(t) \in \mathrm{cl}F \setminus F\} \tag{4}$$

has a Lebesgue measure $\lambda(\Delta)$ less than δ ;

(iii) $z(t) = \gamma(t)$, for all $t \in \Delta \cup \{0,1\}$.

Proof. Since one-dimensional bounded subanalytic sets are just finite unions of intervals (see [6] e.g.), the result is obvious for $n = 1$. So let us assume that $n \geq 2$ and let us set

$$M := \gamma([0,1]).$$

With no loss of generality we may also assume that

$$M \subset \mathrm{cl}F \setminus F \tag{5}$$

and that $M' := \gamma((0,1))$ is a subanalytic C^∞-submanifold of $\mathbb{R}^n$. By Proposition 4 there exists a subanalytic stratification $\mathcal{P}$ of the subanalytic set $F \cup M$ compatible with the family $\{F, M\}$. Then F is a union of a finite subfamily of $\mathcal{P}$, thus, in view of (5) and properties (iii) and (iv) of Proposition 4, there exist $\{C_{\ell_1}, \ldots, C_{\ell_k}\} \subset \mathcal{P}$, with $\dim C_{\ell_i} \geq 2$ and

$$M \subset \bigcup_{i=1}^{k} (\mathrm{cl}C_{\ell_i} \setminus C_{\ell_i}).$$

Since the strata are finite and disjoint, it is clearly enough to restrict ourselves to the case that

$$M \subset \mathrm{cl}C_j \setminus C_j,$$

where C_j is a stratum of $F \cup M$ of dimension greater or equal to 2 entirely included in F. Resorting to the wing's lemma (e.g. [11], [9]), the stratification $\{C_i\}_{i=1}^p$ can be refined in such a way that there exists a stratum C_j satisfying $\dim C_j = \dim M + 1 = 2$. We may also identify M' to $(0,1) \times \{0\}^{n-1}$, so that there exists a continuous subanalytic mapping $\varphi : M' \times [0,1) \to \mathbb{R}^{n-2}$ whose restriction to $M' \times (0,1)$ is analytic and such that

$$\varphi(s,0) = 0_{n-2}, \quad \text{for all } s \in M', \tag{6}$$

(0_{n-2} denotes the zero of $\mathbb{R}^{n-2}$) and

$$C_j = \{(s,\tau,w) \in M' \times (0,1) \times \mathbb{R}^{n-2} : w = \varphi(s,\tau)\}.$$

With this notation, let us write

$$\gamma(t) = (s(t), 0, 0_{n-2}), \qquad \text{for all } t \in [0,1]. \tag{7}$$

Applying Pawlucki's version of the Puiseux theorem ([16, Proposition 2]), we obtain for every $\bar{s} \in M$ a neighborhood $B_{\bar{s}}$ of $\bar{s}$, $\delta_0 > 0$, an integer $r > 0$, a finite subset N of $M_{\bar{s}} := M \cap B_s$, and an analytic function

$$\psi : M_{\bar{s}} \diagdown N \times (-\delta_0, \delta_0) \to \mathbb{R}^{n-2},$$

such that

$$\psi(s,\tau) = \varphi(s,\tau^r), \qquad \text{for all } (s,\tau) \in M_{\bar{s}} \diagdown N \times (0,\delta_0). \tag{8}$$

Since M is compact, a standard argument shows that assuming $M_{\bar{s}} = M$ does not restrict generality.

Let further $t_1 < \ldots < t_p$ in $[0,1]$ be such that $N = \{s(t_i) : 1 \le i \le p\}$. It suffices to prove the result for the case $p = 2$. Fix $\delta > 0$, choose $\varepsilon \in (0, \min\{\frac{t_2 - t_1}{2}, \frac{\delta}{2}\})$ and consider any subanalytic function $u : [t_1, t_2] \to [0, \varepsilon]$ which is C^1 on the interval $(t_1 + \varepsilon, t_2 - \varepsilon)$ and has the properties:

- $u(t) = 0$, for all $t \in [t_1, t_1 + \varepsilon] \cup [t_2 - \varepsilon, t_2]$
- $u(t) > 0$ and $|\dot{u}(t)| < \varepsilon$, for all $t \in (t_1 + \varepsilon, t_2 - \varepsilon)$.

We now define

$$z(t) = (s(t), u(t), \psi(s(t), u(t))), \qquad \text{for all } t \in [t_1, t_2]. \tag{9}$$

It follows directly from (6), (7) and (8) that

$$\gamma(t) = (s(t), 0, \psi(s(t), 0)), \qquad \text{for all } t \in [t_1, t_2].$$

Since u is positive on $(t_1 + \varepsilon, t_2 - \varepsilon)$, it follows that $t \notin \Delta$ whenever $z(t) \neq \gamma(t)$, so (iii) holds. Assertion (ii) follows from the choice of (a small) $\varepsilon > 0$. To prove assertion (i), let us note that for all $t \in \Delta$ we have $u(t) = 0$ and $z(t) = \gamma(t)$. It follows that $\dot{z}(t) = \dot{\gamma}(t)$ for all but finitely many $t \in \Delta$. On the other hand, (6) and (8) imply that $\frac{d}{dt}[\psi(s(t),0)] = 0$, for all $t \in [t_1, t_2]$. Noting that ψ is analytic around $\gamma((t_1 + \varepsilon, t_2 - \varepsilon)) \times (-\delta_0, \delta_0)$, shrinking ε if necessary, and using the properties of u we see that (i) holds. This completes the proof. $\diamond$

We are ready to state the first main result of the section.

Theorem 13 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a globally subanalytic function such that $f|_{\operatorname{dom} f}$ is continuous with $\operatorname{dom} f$ closed. Then f is constant on each connected component of the set L-$\operatorname{crit} f$ of its lower critical points.*

Proof. Combining Proposition 10 with Proposition 3 we infer that the set L-$\operatorname{crit} f$ has a finite number of connected components. Let S denote any of these connected components and consider any two points x, y in S. Let us prove that $f(x) = f(y)$. By Proposition 3 there exists a continuous subanalytic path

$\gamma : [0,1] \to S$ joining x to y, which we may clearly assume one-to-one. By Lemma 5 the mapping γ is absolutely continuous, so

$$M_1 := \int_0^1 \|\dot{\gamma}(t)\| \, dt \tag{10}$$

is a finite nonnegative number. Let us now consider any $\varepsilon > 0$, and let us define

$$F = \{ x \in \mathbb{R}^n : \exists x^* \in \hat{\partial} f(x), \, \|x^*\| < \varepsilon \}. \tag{11}$$

It follows from Remark 1 that $S \subset \mathrm{cl}F$, and by Proposition 10 that F is globally subanalytic. Set $h := f \circ \gamma$ and let $\delta > 0$ be such that

$$\int_\Delta |\dot{h}(t)| dt < \varepsilon \tag{12}$$

whenever $\Delta \subset [0,1]$ is a finite union of intervals with total length less than δ. (This is possible in view of Lemma 5.)

Let us apply Lemma 12 for the (nonempty) globally subanalytic set F, the subanalytic path γ and the above $\delta > 0$. We obtain a subanalytic path $z : [0,1] \to \mathrm{cl}F$ satisfying properties (i)-(iii) of Lemma 12. In particular, the set Δ defined by (4) is a subanalytic subset of $[0,1]$, thus it is a finite union of intervals (some of them possibly points), whose total length is less than $\delta > 0$. Set $h_1 = f \circ z$. The subanalytic functions $z(t)$ and $h_1(t)$ are differentiable at every $t \in [0,1] \diagdown N$, where N is a finite subset of $[0,1]$. Thus, by Proposition 9 and Remark 3, for every $t \in [0,1] \diagdown (N \cup \Delta)$ we have

$$\emptyset \neq \langle \dot{z}(t), \hat{\partial} f(z(t)) \rangle \subset \hat{\partial} h_1(t) = \{\dot{h}_1(t)\}. \tag{13}$$

Combining (13) with (11) we get

$$|\dot{h}_1(t)| \le \varepsilon \|\dot{z}(t)\|, \qquad \text{for all } t \in [0,1] \diagdown (N \cup \Delta),$$

which in view of (10) and Lemma 12 (i) yields

$$\int_{[0,1] \diagdown (N \cup \Delta)} |\dot{h}_1(t)| \, dt \le \varepsilon \int_{[0,1] \diagdown (N \cup \Delta)} \|\dot{z}(t)\| \, dt \le \varepsilon \, (\delta + M_1) \, . \tag{14}$$

Since N is finite and $h(t) = h_1(t)$ for all $t \in \Delta$, it follows from (12) that

$$\int_{N \cup \Delta} |\dot{h}_1(t)| \, dt = \int_\Delta |\dot{h}(t)| \, dt < \varepsilon \, . \tag{15}$$

Combining (14) and (15) we have

$$|f(x) - f(y)| \le \int_0^1 |\dot{h}_1(t)| \, dt \le \varepsilon \, (\delta + M_1) + \varepsilon \, .$$

Since the last equality holds for every $\varepsilon > 0$, it follows that $f(x) = f(y)$. $\diamond$

The main result of this section is now obtained as a consequence of the above result.

Theorem 14 (generalized Morse-Sard theorem) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a subanalytic function such that $f\,|_{\mathrm{dom}\, f}$ is continuous with* $\mathrm{dom}\, f$ *closed. Then*

(i) f is constant on each connected component of the set L-$\mathrm{crit}\, f$ of its lower critical points.

(ii) The set of lower critical values $f(L\text{-}\mathrm{crit}\, f)$ is countable.

Proof. For any $r > 0$ let us denote by B_r the ball of center 0 and radius $r > 0$, and let us define the indicator function $\delta_{B_r} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting $\delta_{B_r}(x) = 0$, if $x \in B_r$ and $+\infty$ otherwise. We also define $g_r = f + \delta_{B_r}$. Then for every $r > 0$ the functions g_r are globally subanalytic and coincide with f on B_r. Moreover the set of their critical points coincide with that of f on the interior of B_r. The first assertion follows directly by applying Theorem 13 to the globally subanalytic functions g_r, for every $r > 0$. Assertion (ii) is now a direct consequence of (i) and the fact that $L\text{-}\operatorname{crit} f$ has a locally finite number of connected components (Proposition 3). $\diamond$

Let us finally state the following result, bootstrapping with the generalized Łojasiewicz inequality for nonsmooth functions established in [4, Theorem 3.5].

Theorem 15 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ subanalytic function such that $f\,|_{\operatorname{dom} f}$ is continuous with $\operatorname{dom} f$ closed. The following assertions are equivalent:*

(i) f has the Sard property (that is f is constant on each connected component of $L\text{-}\operatorname{crit} f$).

(ii) For every $a \in L\text{-}\operatorname{crit} f$ there exist $\delta, \rho > 0$ and an exponent $\theta \in [0, 1)$ such that

$$|f(x) - f(a)|^{\theta} \le \rho \|x^*\| \tag{16}$$

for all $x \in B(a, \delta)$ and every $x^* \in \partial f(x)$.

Proof. (i)$\Longrightarrow$(ii). It follows from [4, Theorem 3.5].

(ii)$\Longrightarrow$(i). This is an obvious consequence of (16) and Definition 7. $\diamond$

Remark 6 If f is a finite-valued continuous subanalytic function, then both Theorem 14 and Theorem 15 can be reformulated in terms of the generalized critical points introduced in Remark 2. More precisely we can assert that f is constant on each connected component of the set of its critical points $\operatorname{crit} f$. Indeed, since every subanalytic path of the set $\operatorname{crit} f$ of critical points can be broken into a sequence of subpaths, consisting of all lower or all upper critical points, the assertion follows by applying Theorem 14 for f and $-f$. As a consequence, we obtain the following refined Łojasiewicz inequality:

(Generalized Łojasiewicz inequality) For every $a \in \operatorname{crit} f$ there exist $\delta, \rho > 0$ and an exponent $\theta \in [0, 1)$ such that

$$|f(x) - f(a)|^{\theta} \le \rho \|x^*\| \tag{17}$$

for all $x \in B(a, \delta)$ and every $x^* \in \partial f(x) \cup -\partial(-f)(x)$.

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