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The Structured Distance to Ill-Posedness for Conic Systems

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An important measure of conditioning of a conic linear system is the size of the smallest structured perturbation making the system ill-posed. We show that this measure is unchanged if we restrict to perturbations of low rank. We thereby derive a broad generalization of the classic Eckart-Young result characterizing the distance to ill-posedness for a linear map.

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1. Introduction. Consider two finite-dimensional normed spaces X and Y, a fixed convex cone $K \subset X$, and a linear mapping A: $X \to Y$. We call A *well-posed* if $AK = Y$. In particular, in the purely linear case $K = X$, well-posedness coincides with surjectivity. Our interest is in the "distance to ill-posedness"; that is, we seek the smallest structured linear perturbation ΔA: $X \to Y$ such that the perturbed mapping $A + \Delta A$ is not well-posed. When $K = X$ and the structure of perturbations is unrestricted, the classical Eckart-Young theorem identifies the distance to ill-posedness as the smallest singular value of A.

For more general convex cones K and unstructured perturbations, seminal work of Renegar (1995a, b) relates the distance to ill-posedness to the complexity of solving associated linear programs. Imposing structure on the allowable perturbations (for example, to maintain a sparsity pattern in the map A) leads to a considerably more involved theory. In the purely linear case $K = X$, such questions arise as "structured singular value" calculations in the area of control theory, pioneered by Doyle and known as "μ-analysis" (Doyle 1982).

In this paper, we follow quite closely the approach of Peña (2003a) in considering structured perturbations to general conic systems. We depend heavily on the same rank-one reduction technique used in Peña (2003a) and introduced in Peña (1998, 2000). Our approach differs in several respects. First, we develop the theory in the concise and elegant language of sublinear set-valued mappings (in other words, mappings the graphs of which are convex cones). This notion lucidly captures the idea of a conic convex system: Well-posedness becomes the notion of surjectivity of the mapping. In this framework, the unstructured case was developed in Lewis (1999) and generalized in Dontchev et al. (2003). (Conversely, although less obviously, conic linear systems can model sublinear mappings, Peña 2003b.)

Secondly, the structured perturbations we consider are rather general, being of the form $\sum_i P_i T_i Q_i$ for linear mappings T_i (where the linear mappings P_i and Q_i are fixed at the outset). Thirdly, we allow arbitrary norms on the underlying spaces. Lastly, our proofs consist of direct duality arguments, avoiding the necessity of "lifting" problems into higher dimensional spaces. In this manner we hope to illuminate the structural simplicity of the key results.

The main result is as follows. We consider finite-dimensional normed spaces X, Y, U_i, and V_i, linear mappings P_i: $V_i \to Y$ and Q_i: $X \to U_i$ (for $i = 1, 2, \ldots, k$), and a surjective set-valued mapping F: $X \rightrightarrows Y$ with graph a closed convex *cone*. Then, denoting dual spaces and adjoint mappings by $*$, the following four quantities are equal:

$$\min_{\text{linear } T_i} \left\{ \max_i \|T_i\|: \ F + \sum_i P_i T_i Q_i \text{ nonsurjective} \right\},$$

$$\min_{\text{rank-one linear } T_i} \left\{ \max_i \|T_i\|: \ F + \sum_i P_i T_i Q_i \text{ nonsurjective} \right\},$$

$$\min_{u_i^* \in U_i^*,\, z_i \geq 0,\, 0 \neq y^* \in Y^*} \left\{ \max_i \frac{z_i}{\|P_i^* y_i\|}: \ \sum_i z_i Q_i^* u_i^* \in F^*(y^*),\ \|u_i^*\| \leq 1 \right\},$$

$$\min_{v_i \in V_i,\, \|v_i\| \leq 1} \sup_{x \in X,\, w_i > 0} \left\{ \min_i \frac{w_i}{\|Q_i x\|}: \ \sum_i w_i P_i v_i \in F(x) \right\}.$$

Throughout, we consider the zero mapping as being *rank-one*. Some parallel results, obtained independently, have recently been reported in Peña (2003c).

2. Rank-one perturbation. As observed by Peña (1998, 2000), the idea of rank-one perturbation is fundamental to the theory of the distance to ill-posedness. Our first, elementary result tries to capture the underlying idea in a way that extends to structured perturbations.

Throughout this article we follow the terminology of Rockafellar and Wets (1998). We call a set-valued mapping $F\colon X \rightrightarrows Y$ *positively homogeneous* if its graph

$$\operatorname{gph} F = \{(x, y) \in X \times Y: \ y \in F(x)\}$$

is a cone (which is to say, nonempty and closed under nonnegative scalar multiplication). To recapture the theory of conic linear systems we typically consider examples of the form

$$F(x) = \begin{cases} \{Ax\} & (x \in K), \\ \varnothing & (x \notin K), \end{cases}$$

where the mapping $A\colon X \to Y$ is linear and $K \subset X$ is a convex cone. The *inverse* of a set-valued mapping F is the mapping $F^{-1}\colon Y \rightrightarrows X$ defined by

$$x \in F^{-1}(y) \Leftrightarrow y \in F(x).$$

We call F *singular* if $F^{-1}(0) \neq \{0\}$.

We typically denote the norm on a normed space X by $\|\cdot\|$ (or by $\|\cdot\|_X$ if we wish to be specific) and the closed unit ball in X by B_X, and we denote the space of linear mappings from X to Y by $L(X, Y)$. In particular, for a mapping $A \in L(X, Y)$, we denote the usual operator norm by $\|A\|$. We denote the dual space of X by X^*, and we write the action of a linear functional $x^* \in X^*$ on an element $x \in X$ as $\langle x^*, x\rangle$. We are particularly interested in rank-one mappings in $L(X, Y)$, which are those mappings of the form $X \ni x \mapsto \langle x^*, x\rangle y$ for some given elements $x^* \in X^*$ and $y \in Y$: we denote the set of such mappings by $L_1(X, Y)$. The norm of this mapping is just $\|x^*\| \cdot \|y\|$.

In what follows, we interpret $1/0 = +\infty$ and $1/{+\infty} = 0$.

Theorem 2.1 (Rank-One Reduction). *Consider finite-dimensional normed spaces X, Y, U, and V, a positively homogeneous set-valued mapping $F\colon X \rightrightarrows Y$, and linear mappings $P\colon V \to Y$ and $Q\colon X \to U$. Then, the quantity in $[0, +\infty]$ defined by*

$$\alpha = \inf_{T \in L(U, V)} \{\|T\|: \ F + PTQ \text{ singular}\}$$

is unchanged if we further restrict the infimum to be over mappings T of rank one. Furthermore, if we assume

$$0 \in F(x) \text{ and } x \neq 0 \quad \Rightarrow \quad Qx \neq 0$$

(*as holds in particular if Q is injective or F is nonsingular*), *then*

$$\frac{1}{\alpha} = \sup_{x \in X,\, v \in B_V} \{\|Qx\|: \ Pv \in F(x)\}.$$

Note. We address the question of the attainment in the above infimum and supremum in the next section.

Proof. Denote the right-hand side of the last equation by β. First, consider the case where F is singular. In this case, clearly $\alpha = 0$ and is attained by choosing the rank-one mapping $T = 0$. Choose any nonzero $x_1 \in F^{-1}(0)$, by assumption $Qx_1 \neq 0$. Now by choosing $x = \lambda x_1$ with $\lambda \in \mathbf{R}_+$ and $v = 0$ in the definition of β and letting λ grow, we see $\beta = +\infty$, so the result holds. We can therefore assume that F is nonsingular.

We next show $\alpha \geq 1/\beta$. Consider any feasible mapping T in the definition of α so there exists a nonzero vector $x \in (F + PTQ)^{-1}(0)$. Hence, we have $-PTQx \in F(x)$; because $F^{-1}(0) = \{0\}$, we deduce $TQx \neq 0$. Positive homogeneity now implies

$$P\left(-\frac{1}{\|TQx\|}TQx\right) \in F\left(\frac{1}{\|TQx\|}x\right),$$

so by definition,

$$\beta \geq \left\|Q\|TQx\|^{-1}x\right\| \geq \frac{1}{\|T\|}.$$

Thus, all feasible T satisfy $\|T\| \geq 1/\beta$, and we deduce $\alpha \geq 1/\beta$.

Next, we define the quantity

$$\gamma = \inf_{T \in L_1(U,V)} \{\|T\|: F + PTQ \text{ singular}\}.$$

Clearly, we have the inequality $\gamma \geq \alpha$, so it now suffices to prove $\gamma \leq 1/\beta$. If $\beta = 0$, there is nothing to prove, so we can assume $\beta > 0$.

Consider any feasible vectors x and v in the definition of β. Because $\beta > 0$, we can assume $Qx \neq 0$. There exists a norm-one linear functional $u^* \in U^*$ satisfying $\langle u^*, Qx \rangle = \|Qx\|$. Now we have

$$0 \in F(x) - Pv = F(x) - PTQx,$$

where $T: U \to V$ is the rank-one linear map defined by

$$Tu = \frac{\langle u^*, u \rangle}{\|Qx\|} v.$$

Because we know that $\|u\|_* = 1$ and $\|v\| \leq 1$, we deduce

$$\gamma \leq \|T\| \leq \frac{1}{\|Qx\|},$$

so $1/\gamma \geq \|Qx\|$. Finally, taking the supremum over all feasible vectors x and v in the definition of β shows $1/\gamma \geq \beta$, as required. □

Note that if $X = Y$, the mapping F is single-valued and linear, and the mappings P and Q are just the identity, then we recover the classical Eckart-Young theorem.

We next generalize to perturbations with a composite structure. In conformity with our previous usage, for $z \in \mathbf{R}_+$ we define

$$\frac{z}{0} = \begin{cases} +\infty & (z > 0), \\ 0 & (z = 0). \end{cases}$$

Corollary 2.2 (Rank-One Reduction for Sums). *Given finite-dimensional normed spaces X, Y, U_i, and V_i, a positively-homogeneous set-valued mapping $F: X \rightrightarrows Y$, and linear mappings $P_i: V_i \to Y$ and $Q_i: X \to U_i$ (for $i = 1, 2, \ldots, k$), the quantity*

$$\alpha := \inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\|: F + \sum_i P_i T_i Q_i \ \textit{singular} \right\}$$

is unchanged if we further restrict the infimum to be over mappings T_i of rank one. Consequently, we have the following:

$$\begin{aligned}\alpha &= \inf_{v_i\in B_{V_i},\, z_i\in \mathbf{R}_+,\, 0\neq x\in X}\left\{\max_i \frac{z_i}{\|Q_i x\|}\colon\ \sum_i z_i P_i v_i \in F(x)\right\}\\ &= \inf_{v_i\in V_i,\, u_i^*\in B_{U_i^*},\, 0\neq x\in X}\left\{\max_i \|v_i\|\colon\ \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(-x),\ \langle u_i^*, Q_i x\rangle \geq 0\ \forall\, i\right\}.\end{aligned}$$

Note. As before, we address the question of the attainment in the above infima in the next section.

Proof. Fix any real $\epsilon > 0$ and consider any feasible mappings T_i in the above infimum. By applying the preceding theorem we see that there exists a mapping $\widehat{T}_k \in L_1(U_k, V_k)$ satisfying $\|\widehat{T}_k\| < \|T_k\| + \epsilon$ and

$$\left(F + \sum_{i=1}^{k-1} P_i T_i Q_i + P_k \widehat{T}_k Q_k\right)^{-1}(0) \neq \{0\}.$$

We can continue in this fashion, arriving at mappings $\widehat{T}_i \in L_1(U_i, V_i)$ satisfying $\|\widehat{T}_i\| < \|T_i\| + \epsilon$ (for $i = 1, 2, \ldots, k$) and

$$\left(F + \sum_i P_i \widehat{T}_i Q_i\right)^{-1}(0) \neq \{0\}.$$

Because $\epsilon > 0$ was arbitrary, the rank-one reduction now follows.

Consequently, we have $\alpha = \alpha_1$, where

$$\begin{aligned}\alpha_1 &:= \inf_{v_i\in V_i,\, u_i^*\in U_i^*,\, 0\neq x\in X}\left\{\max_i \|v_i\|\|u_i^*\|\colon\ \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(x)\right\}\\ &\leq \inf_{v_i\in B_{V_i},\, u_i^*\in U_i^*,\, 0\neq x\in X}\left\{\max_i \|v_i\|\|u_i^*\|\colon\ \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(x)\right\}\\ &\leq \alpha_2,\end{aligned}$$

where

$$\alpha_2 := \inf_{v_i\in B_{V_i},\, u_i^*\in U_i^*,\, 0\neq x\in X}\left\{\max_i \|u_i^*\|\colon\ \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(x)\right\}.$$

On the other hand, suppose that the vectors v_i, u_i^*, and x are feasible in the infimum defining α_1. If we define, for each index i,

$$(\hat{v}_i, \hat{u}_i^*) = \begin{cases} (\|v_i\|^{-1} v_i,\ \|v_i\|u_i^*) & (v_i \neq 0),\\ (0, 0) & (v_i = 0),\end{cases}$$

then the vectors $\hat{v}_i$, $\hat{u}_i^*$, and x are feasible in the infimum defining α_2, and $\|\hat{u}_i^*\| = \|v_i\|\|u_i^*\|$ for each i. This proves $\alpha_2 \leq \alpha_1$, so in fact $\alpha = \alpha_1 = \alpha_2$.

A completely analogous argument shows

$$\alpha = \inf_{v_i\in V_i,\, u_i^*\in B_{U_i^*},\, 0\neq x\in X}\left\{\max_i \|v_i\|\colon\ \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(-x)\right\}.$$

The final expression for α claimed in the theorem now follows, because the additional conditions $\langle u_i^*, Q_i x\rangle \geq 0$ impose no essential restriction: For any index i we can always replace the pair of vectors (v_i, u_i^*) with $(-v_i, -u_i^*)$ without changing feasibility or the objective value.

Considering the definition of α_2, we observe for any vectors v_i,

$$\inf_{u_i^* \in U_i^*,\, 0\neq x\in X} \left\{ \max_i \|u_i^*\| \colon \sum_i \langle u_i^*, Q_i x\rangle P_i v_i \in F(x) \right\}$$
$$= \inf_{u_i^* \in U_i^*,\, 0\neq x\in X,\, z_i\in \mathbf{R}_+} \left\{ \max_i \|u_i^*\| \colon \sum_i z_i P_i v_i \in F(x),\, \langle u_i^*, Q_i x\rangle = z_i, \right\}$$

because a feasible choice of the variables on the right-hand side immediately gives a feasible choice on the left-hand side with the same objective value, while for any feasible choice of vectors u_i^* and x on the left-hand side, setting $\hat{u}_i^* = (\operatorname{sgn}\langle u_i^*, Q_i x\rangle) u_i^*$ and $\hat{z}_i = |\langle u_i^*, Q_i x\rangle|$ for each index i gives a feasible choice on the right-hand side with the same objective value.

By observing that, for any vector $x \in X$ and scalar $z_i \in \mathbf{R}_+$, we have

$$\inf_{u_i^* \in U_i^*} \{\|u_i^*\| \colon \langle u_i^*, Q_i x\rangle = z_i\} = \frac{z_i}{\|Q_i x\|},$$

and the result follows. □

Note. It is not hard to see that the case $k = 1$ gives back Theorem 2.1.

3. Duality and surjectivity. We return to our motivating example of the well-posedness of a linear mapping $A\colon X \to Y$ relative to a convex cone $K \subset X$ (by which we mean $AK = Y$). If, as before, we define an associated set-valued mapping $F\colon X \rightrightarrows Y$ by

$$F(x) = \begin{cases} \{Ax\} & (x \in K), \\ \varnothing & (x \notin K), \end{cases} \tag{3.1}$$

then well-posedness holds exactly when $F(X) = Y$.

We call a general set-valued mapping $F\colon X \rightrightarrows Y$ *surjective* if $F(X) = Y$, *closed* if its graph is closed, and *sublinear* if its graph is a convex cone. Sublinear set-valued mappings are also known as *convex processes*. The notions of singularity and surjectiveness are intimately connected via duality: the *adjoint* of F is the set-valued mapping $F^*\colon Y^* \to X^*$ defined by

$$x^* \in F^*(y^*) \quad \Leftrightarrow \quad \langle y^*, y\rangle \geq \langle x^*, x\rangle \text{ whenever } y \in F(x).$$

The adjoint is easily seen to be closed and sublinear and coincides with the classic notion for single-valued linear mappings. More generally, direct calculation shows that for any linear mapping $G\colon X \to Y$ we have $(F+G)^* = F^* + G^*$. It is simple to check that the adjoint of the set-valued mapping (3.1) is defined by $F^*(y^*) = A^* y^* + K^*$, where $K^* \subset X^*$ is the usual (negative) polar cone for K.

The relationship between surjectiveness and singularity is described by the following concise result, a special case of an infinite-dimensional version of the open mapping theorem (Borwein 1986).

Theorem 3.2 (Open Mapping). *For finite-dimensional normed spaces X and Y, a closed sublinear set-valued mapping $F\colon X \rightrightarrows Y$ is surjective if and only if its adjoint mapping F^* is nonsingular.*

Note 3.3. If the closed sublinear set-valued mapping F is surjective, then so is the mapping $F + G$ for all small linear mappings G, and the analogous result also holds for nonsingularity (Robinson 1976). Hence, with this assumption on F in Theorem 2.1 (rank-one reduction), the infimum

$$\inf_{T \in L(U,V)} \{\|T\| \colon F + PTQ \text{ singular}\}$$

is attained whenever finite, because it seeks the norm of the smallest element in a nonempty closed set. In this case, following the proof shows both the same infimum over the rank-one mappings T, and the supremum

$$\sup_{x\in X,\, v\in B_V} \{\|Qx\|\colon Pv \in F(x)\}$$

are also attained.

Note 3.4. Using the preceding note, if the closed sublinear set-valued mapping F is surjective in Corollary 2.2 (rank-one reduction for sums), then the infimum

$$\inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\| \colon F + \sum_i P_i T_i Q_i \text{ singular} \right\}$$

is attained whenever finite, whether over general or rank-one linear mappings T_i, and in this case the infimum

$$\inf_{v_i \in B_{V_i},\, z_i \in \mathbf{R}_+,\, 0\neq x \in X} \left\{ \max_i \frac{z_i}{\|Q_i x\|} \colon \sum_i z_i P_i v_i \in F(x) \right\}$$

is also attained.

Using Theorem 3.2 (open mapping), we can quickly derive a version of Corollary 2.2 (rank-one reduction for sums) for nonsurjectivity rather than singularity.

Theorem 3.5 (Rank Reduction and Surjectivity). *For any finite-dimensional normed spaces X, Y, U_i, and V_i, closed sublinear set-valued mapping $F\colon X \rightrightarrows Y$, and linear mappings $P_i\colon V_i \to Y$ and $Q_i\colon X \to U_i$ (for $i = 1, 2, \ldots, k$), the quantity*

$$\alpha := \inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\| \colon F + \sum_i P_i T_i Q_i \textit{ nonsurjective} \right\}$$

is unchanged if we further restrict the infimum to be over mappings T_i of rank one, and in fact

$$\begin{aligned}\alpha &= \inf_{u_i^* \in B_{U_i^*},\, z_i \in \mathbf{R}_+,\, 0\neq y^* \in Y^*} \left\{ \max_i \frac{z_i}{\|P_i^* y^*\|} \colon \sum_i z_i Q_i^* u_i^* \in F^*(y^*) \right\} \\ &= \inf_{v_i \in B_{V_i},\, u_i^* \in U_i^*,\, 0\neq y^* \in Y^*} \left\{ \max_i \|u_i^*\| \colon \sum_i \langle y^*, P_i v_i\rangle Q_i^* u_i^* \in F^*(-y^*),\ \langle y^*, P_i v_i\rangle \geq 0\ \forall\, i \right\}.\end{aligned}$$

Furthermore, all four infima are attained if α is finite.

Proof. By the open mapping theorem, we have

$$\begin{aligned}\alpha &= \inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\| \colon \left(F + \sum_i P_i T_i Q_i\right)^* \text{ singular} \right\} \\ &= \inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i^*\| \colon F^* + \sum_i Q_i^* T_i^* P_i^* \text{ singular} \right\},\end{aligned}$$

because the adjoint transformation $*\colon L(U_i, V_i) \to L(V_i^*, U_i^*)$ leaves the norm fixed. This transformation is in fact a bijection, which also preserves the classes of rank-one mappings. Corollary 2.2 (rank-one reduction for sums) ensures that the infimum is unchanged if we restrict to mappings T_i for which T_i^* is rank-one, or, in other words, to rank-one T_i, as required. The final expressions follow directly from Corollary 2.2. The final claim concerning attainment follows from Note 3.4. □

4. Duality. Our ultimate aim is to express the structured distance to nonsurjectivity in terms involving the mapping F rather than its adjoint. For this purpose, the following result is crucial.

Theorem 4.1 (Theorem of the Alternative). *For any finite-dimensional normed spaces X, Y, and U_i, surjective closed sublinear set-valued mapping $F\colon X \rightrightarrows Y$, linear map-*

pings Q_i: $X \to U_i$, and vectors $y_i \in Y$ (for $i = 1, 2, \ldots, k$), exactly one of the following two systems has a solution:

(i) $\sum_i w_i y_i \in F(x)$, $\|Q_i x\| < w_i \in \mathbf{R}$ *for each* i, $x \in X$,

(ii) $\sum_i \langle y^*, y_i\rangle Q_i^* u_i^* \in F^*(-y^*)$, $0 \neq y^* \in Y^*$, $\langle y^*, y_i\rangle \geq 0$ *and* $u_i^* \in B_U$ *for each* i.

Proof. Suppose first that both systems have solutions. By the definition of the adjoint, we deduce the inequality

$$\left\langle -y^*, \sum_i w_i y_i \right\rangle \geq \left\langle \sum_i \langle y^*, y_i\rangle Q_i^* u_i^*, x \right\rangle$$

or, equivalently,

$$0 \geq \sum_i \langle y^*, y_i\rangle (w_i + \langle u_i^*, Q_i x\rangle).$$

Now each term in the sum on the right-hand side is a product of two factors, the first of which is nonnegative and the second of which is strictly positive. Hence, this inequality can hold only if $\langle y^*, y_i\rangle = 0$ for each index i, and in this case we deduce $0 \in F^*(-y^*)$. But the mapping F is surjective, so by Theorem 3.2 (open mapping) its adjoint F^* is nonsingular, and this is a contradiction. Hence, at most one of the two systems has a solution.

Suppose now that system (i) has no solution. Then, the two convex subsets of $X \times \mathbf{R}^k$,

$$\left\{(x, w)\colon \sum_i w_i y_i \in F(x)\right\} \qquad \text{and} \qquad \{(x, w)\colon \|Q_i x\| < w_i \text{ for each } i\},$$

are disjoint. Both sets are clearly nonempty, so there exists a separating hyperplane: There exists a nonzero vector $(x^*, w^*) \in X^* \times R^k$ and a real μ such that the two implications

$$\sum_i w_i y_i \in F(x) \quad \Rightarrow \quad \langle x^*, x\rangle - \sum_i w_i^* w_i \geq \mu \qquad \text{and}$$

$$\|Q_i x\| < w_i \text{ for each } i \quad \Rightarrow \quad \langle x^*, x\rangle - \sum_i w_i^* w_i \leq \mu$$

hold.

Considering the first implication, by the positive homogeneity of F, we deduce

$$\sum_i w_i y_i \in F(x) \quad \Rightarrow \quad \langle x^*, x\rangle - \sum_i w_i^* w_i \geq 0 \tag{4.2}$$

and $\mu \leq 0$. This, in conjunction with the second implication, shows that

$$w_i^* \geq 0 \quad \text{for each } i \tag{4.3}$$

and

$$\langle x^*, x\rangle \leq \sum_i w_i^* \|Q_i x\| \quad \text{for all } x \in X.$$

This inequality expresses the fact that the vector x^* is a subgradient at the origin for the convex function

$$x \mapsto \sum_i w_i^* \|Q_i x\|,$$

so by standard convex analysis we deduce

$$x^* \in \sum_i w_i^* Q_i^* B_{U_i^*}. \tag{4.4}$$

We now apply a rather standard duality argument to implication (4.2). We define a function $f\colon Y \to [-\infty, +\infty]$ by

$$f(y) = \inf_{x\in X,\, w_i\in\mathbf{R}} \left\{ \langle x^*, x\rangle - \sum_i w_i^* w_i \colon y + \sum_i w_i y_i \in F(x) \right\}.$$

Implication (4.2) shows that $f(0) = 0$, and a standard elementary argument using the convexity of the graph of F shows that f is convex. Because the mapping F is surjective, the function f never takes the value $+\infty$. Consequently (see Rockafellar 1970), f has a subgradient $y^* \in Y^*$ at the origin, or in other words,

$$y + \sum_i w_i y_i \in F(x) \quad \Rightarrow \quad \langle y^*, y\rangle \le \langle x^*, x\rangle - \sum_i w_i^* w_i.$$

Setting $x = 0$ and $y = -\sum_i w_i y_i$ shows that

$$\sum_i w_i (w_i^* - \langle y^*, y_i\rangle) \le 0 \quad \text{for all } w \in \mathbf{R}^k,$$

so

$$w_i^* = \langle y^*, y_i\rangle \quad \text{for each } i. \tag{4.5}$$

Furthermore, setting each $w_i = 0$ shows that

$$y \in F(x) \quad \Rightarrow \quad \langle y^*, y\rangle \le \langle x^*, x\rangle,$$

or in other words,

$$-x^* \in F^*(-y^*). \tag{4.6}$$

Finally, putting together the relationships (4.3), (4.4), (4.5), and (4.6) shows that we have constructed a solution to system (ii) in the theorem statement, as required. □

A helpful restatement of the above theorem is contained in the following duality result. Recall our convention $z/0 = +\infty$ for real $z > 0$.

Theorem 4.7 (Duality). *Consider finite-dimensional normed spaces X, Y, and U_i, a surjective closed sublinear set-valued mapping $F\colon X \rightrightarrows Y$, linear mapping $Q_i\colon X \to U_i$, and vectors $y_i \in Y$ (for $i = 1, 2, \ldots, k$). Then, the function $\Phi\colon Y^k \to [0, +\infty]$ defined by*

$$\Phi((y_i)) = \sup_{x\in X,\, 0<w_i\in\mathbf{R}} \left\{ \min_i \frac{w_i}{\|Q_i x\|} \colon \sum_i w_i y_i \in F(x) \right\} \tag{4.8}$$

is lower semicontinuous, and

$$\Phi((y_i)) = \inf_{u_i^*\in U_i^*,\, 0\ne y^*\in Y^*} \left\{ \max_i \|u_i^*\| \colon \sum_i \langle y^*, y_i\rangle Q_i^* u_i^* \in F^*(-y^*),\ \langle y^*, y_i\rangle \ge 0\ \forall\, i \right\}.$$

Furthermore, the infimum on the right-hand side is attained whenever finite.

Proof. We first prove the lower semicontinuity. For each index i consider a sequence of vectors $y_i^r \to y_i$ in the space Y, and consider a sequence of reals $s^r \to s$ as $r \to \infty$ satisfying $s^r \ge \Phi((y_i^r))$, or in other words,

$$x \in X,\ 0 < w_i \in \mathbf{R},\ \text{and } \sum_i w_i y_i^r \in F(x) \quad \Rightarrow \quad s^r \ge \min_i \frac{w_i}{\|Q_i x\|}. \tag{4.9}$$

Consider reals $w_i > 0$ (for each i) satisfying $\sum_i w_i y_i \in F(\bar{x})$. We want to show the inequality

$$s \ge \min_i \frac{w_i}{\|Q_i \bar{x}\|}.$$

To see this, we first note that because F is surjective, it is everywhere *open*: the image under F of any open set is open. In particular, for any real $\delta > 0$, the set $F(\bar{x} + \operatorname{int} \delta B_X)$ is an open neighborhood of the vector $\sum_i w_i y_i$, so for large r must contain the point $\sum_i w_i y_i^r$. Using this tool, we see that there exists a subsequence R of the natural numbers such that

$$\sum_i w_i y_i^r \in F(x^r) \quad \text{for all } r \in R,$$

$$\lim_{r\to\infty,\, r\in R} x^r = \bar{x}.$$

Applying property (4.9) shows

$$s^r \geq \min_i \frac{w_i}{\|Q_i x^r\|} \quad \text{for all } r \in R.$$

Hence, there exists an index $j \in \{1, 2, \ldots, k\}$ and a further subsequence R' of R such that

$$s^r \geq \frac{w_j}{\|Q_j x^r\|} \quad \text{for all } r \in R'.$$

Taking the limit as $r \to \infty$ shows that

$$s \geq \frac{w_j}{\|Q_j \bar{x}\|} \geq \min_i \frac{w_i}{\|Q_i \bar{x}\|},$$

as required. Thus, the function Φ is indeed lower semicontinuous.

Denote the right-hand side of the second claimed expression for Φ by $\Psi((y_i))$: we next want to prove that this infimum is attained whenever $\Psi((y_i))$ is finite. Note that the infimum is unchanged if we add the condition $\|y^*\| = 1$ using positive homogeneity. Now suppose that the infimum is finite, so there exist feasible vectors $\bar{u}_i^*$ and $\bar{y}^*$. If we define $\beta = \max_i \|\bar{u}_i^*\|$, then we can rewrite the infimum as

$$\inf_{u_i^* \in U_i^*,\, y^* \in Y^*} \left\{ \max_i \|u_i^*\| \colon \sum_i \langle y^*, y_i \rangle Q_i^* u_i^* \in F^*(-y^*),\ \|y^*\| = 1,\ \langle y^*, y_i \rangle \geq 0,\ \|u_i^*\| \leq \beta\ \forall\, i \right\}.$$

This quantity involves minimizing a continuous function over a nonempty compact set, so the infimum is attained.

It remains to prove that the two functions Φ and Ψ are identical. Consider any real $\psi > 0$. Using the attainment property we have just proved for Ψ, the statement $\Psi((y_i)) \leq \psi$ is equivalent to the solvability of the system

$$\sum_i \langle y^*, y_i \rangle Q_i^* u_i^* \in F^*(-y^*), \quad 0 \neq y^* \in Y^*,$$
$$\langle y^*, y_i \rangle \geq 0, \quad u_i^* \in \psi B_{U_i^*} \qquad \text{for each } i,$$

or equivalently, to the solvability of the system

$$\sum_i \langle y^*, \psi^{-1} y_i \rangle Q_i^* u_i^* \in F^*(-y^*), \quad 0 \neq y^* \in Y^*,$$
$$\langle y^*, \psi^{-1} y_i \rangle \geq 0, \quad u_i^* \in B_{U_i^*} \qquad \text{for each } i.$$

Using Theorem 4.1 (theorem of the alternative), this is equivalent to the unsolvability of the system

$$\sum_i w_i \psi^{-1} y_i \in F(x), \quad \|Q_i x\| < w_i \in \mathbf{R} \text{ for each } i, \qquad x \in X,$$

or equivalently (because F is positively homogeneous), to the unsolvability of the system

$$\sum_i w_i y_i \in F(x), \quad \psi < \frac{w_i}{\|Q_i x\|},\ 0 < w_i \in \mathbf{R} \text{ for each } i, \qquad x \in X.$$

But this in turn is equivalent to the statement $\Phi((y_i)) \leq \psi$. To summarize, we have shown for all real $\psi > 0$, that

$$\Psi((y_i)) \leq \psi \quad \Leftrightarrow \quad \Phi((y_i)) \leq \psi.$$

The result now follows. $\square$

5. The main result. We now have all the tools we need to derive our main result.

THEOREM 5.1 (DISTANCE TO NONSURJECTIVITY). *For any finite-dimensional normed spaces X, Y, U_i, and V_i, closed sublinear surjective set-valued mapping $F\colon X \rightrightarrows Y$, and linear mappings $P_i\colon V_i \to Y$ and $Q_i\colon X \to U_i$ (for $i = 1, 2, \ldots, k$), the following four quantities are equal*:

$$\inf_{T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\|\colon F + \sum_i P_i T_i Q_i \text{ nonsurjective} \right\},$$

$$\inf_{\text{rank-one } T_i \in L(U_i, V_i)} \left\{ \max_i \|T_i\|\colon F + \sum_i P_i T_i Q_i \text{ nonsurjective} \right\},$$

$$\inf_{u_i^* \in B_{U_i^*},\, z_i \geq 0,\, 0 \neq y^* \in Y^*} \left\{ \max_i \frac{z_i}{\|P_i^* y_i\|}\colon \sum_i z_i Q_i^* u_i^* \in F^*(y^*) \right\},$$

$$\inf_{v_i \in B_{V_i}} \sup_{x \in X,\, w_i > 0} \left\{ \min_i \frac{w_i}{\|Q_i x\|}\colon \sum_i w_i P_i v_i \in F(x) \right\}.$$

Furthermore, if these quantities are finite, each infimum above is attained.

PROOF. The equality of the first three expressions follows immediately from Theorem 3.5 (rank reduction and surjectivity). The last expression also follows from the same result, after applying Theorem 4.7 (duality). □

As pointed out in Peña (2003a), theorems of this type provide an elegant unifying framework for diverse results in the literature (Doyle 1992; Demmel 1982; Rohn 1989; Rump 1997, 1999).

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