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Self-concordant barriers for hyperbolic means

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Abstract. The geometric mean and the function $(\det(\cdot))^{1/m}$ (on the m-by-m positive definite matrices) are examples of "hyperbolic means": functions of the form $p^{1/m}$, where p is a hyperbolic polynomial of degree m. (A homogeneous polynomial p is "hyperbolic" with respect to a vector d if the polynomial $t \mapsto p(x+td)$ has only real roots for every vector x.) Any hyperbolic mean is positively homogeneous and concave (on a suitable domain): we present a self-concordant barrier for its hypograph, with barrier parameter $O(m^2)$. Our approach is direct, and shows, for example, that the function $-m\log(\det(\cdot)-1)$ is an m^2-self-concordant barrier on a natural domain. Such barriers suggest novel interior point approaches to convex programs involving hyperbolic means.

1. Introduction

1. Self-concordant barriers. In 1988, Nesterov and Nemirovskii developed a general, polynomial time framework for convex programming problems, presented in their extensive monograph [8]. This framework for interior point methods relies on the notion of *self-concordant barrier functions*. These functions are special, convex penalty functions which intricately regulate their own behaviour and growth.

We begin by giving the definition of a self-concordant barrier function. Let E be a finite-dimensional real vector space and Q be an open nonempty convex subset of E. A function $F : Q \to \mathbb{R}$ is called a *self-concordant barrier* if it is three times differentiable, convex and satisfies the conditions

$$|D^3F(x)[h,h,h]| \le 2\,(D^2F(x)[h,h])^{3/2}, \tag{1}$$

$$F(x^r) \to \infty \ \text{ for any sequence } x^r \to x \in \partial Q, \text{ and} \tag{2}$$

$$|DF(x)[h]| \le \sqrt{\vartheta}\,(D^2F(x)[h,h])^{1/2}, \tag{3}$$

for all $h \in E$, $x \in Q$. Here $\vartheta \ge 1$ is a fixed constant depending on the function F only, and $D^kF(x)[h,\ldots,h] = \frac{d^k}{dt^k}F(x+th)\Big|_{t=0}$ is the k-th directional derivative at x along the direction h. The constant ϑ is called the *parameter* of the barrier function: smaller parameters ensure that the interior point method using F runs faster. For short we call F a *ϑ-self-concordant barrier*.

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If in addition cl Q is a cone and instead of conditions (1), (2), and (3) the function F satisfies conditions (1), (2), and

$$F(tx) = F(x) - \vartheta \log(t), \text{ for all } x \in Q, \ t > 0, \tag{4}$$

we say F is a *ϑ-normal barrier*. In fact condition (4) implies condition (3), see [8, Corollary 2.3.2].

Note 1. Observe that if F is ϑ-self-concordant then kF is $k\vartheta$-self-concordant for any constant $k \geq 1$.

One of the most important results in Nesterov and Nemirovskii [8] is that a self-concordant barrier function exists for every open convex set Q. They construct such a function, called the *universal barrier*. The parameter ϑ in their construction has magnitude $O(\dim E)$. Because ϑ plays an important role for the convergence speed of the underlying interior point method the question of finding computable barrier functions with small parameters is of fundamental interest.

A crucial example in contemporary optimization is the function $-\log\det(\cdot)$, which is an n-normal barrier for the cone of $n \times n$ symmetric positive definite matrices, a set of dimension $n(n+1)/2$ (see [8]). In this work we show, for example, that $-n\log(\det(\cdot) - 1)$ is a 'shifted' n^2-self-concordant barrier on a corresponding subset of the positive definite cone.

2. Hyperbolic polynomials. Hyperbolic polynomials originate in the theory of partial differential equations and are connected with the well-posedness of the Cauchy problem. For more details about this theory we refer the reader to [4], [6, §12.3–12.6], or for those only interested in a succinct exposition of the main ideas, see [5]. There is recent interest within the optimization community in hyperbolic polynomials because their roots exhibit some nice convexity properties [3], and the polynomials themselves can be used to construct self-concordant barriers with small parameters [5]. We now give the necessary definitions for these polynomials.

A polynomial p on a finite-dimensional real vector space E is *homogeneous of degree m*, if $p(tx) = t^m p(x)$, for all $t \in \mathbb{R}$ and every $x \in E$. For such a polynomial p and a direction $d \in E$ with $p(d) \neq 0$, we say that p is *hyperbolic with respect to d* if the polynomial $t \mapsto p(x + td)$ (where t is a scalar) has only real zeros for every $x \in E$. In this case for every $x \in E$, we can write

$$p(x + td) = p(d) \prod_{i=1}^{m} (t + t_i(x, d)),$$

where $t_i(x, d) = t_i(x)$ are minus the roots of the polynomial $t \mapsto p(x + td)$. Here are a few examples of hyperbolic polynomials.

(a) $E = \mathbb{R}^n$. The polynomial

$$p(x) = \prod_{i=1}^{n} x_i$$

is hyperbolic with respect to the direction $d = (1, ..., 1)$.

(b) $E = \mathbb{R}^n$. The polynomial

$$p(x) = x_1^2 - \sum_{i=2}^{n} x_i^2$$

is hyperbolic with respect to the direction $d = (1, 0, ..., 0)$.

(c) $E = S^n$ (the set of $n \times n$ symmetric matrices). The polynomial

$$p(X) = \det X$$

is hyperbolic with respect to the direction $d = I$.

(d) $E = M_{p,q} \times \mathbb{R}$ (where $M_{p,q}$ is the space of $p \times q$ real matrices, and we assume $q \leq p$). The polynomial

$$p(X, r) = \det\left(X^T X - r^2 I_q\right) \qquad (X \in M_{p,q},\ r \in \mathbb{R})$$

is hyperbolic with respect to the direction $d = (0, 1)$.

We can construct many new hyperbolic polynomials from old ones (see [5], for example). For example if p is hyperbolic of degree m with respect to d and we write it as

$$p(x + td) = \sum_{i=0}^{m} p_i(x) t^i,$$

then each $p_i(x)$ is hyperbolic of degree $m - i$ with respect to d, see [2, p. 130]. Applying this to example (a) gives us that all elementary symmetric polynomials

$$e_k(x) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$$

are hyperbolic of degree k ($k \leq n$) with respect to $d = (1, ..., 1)$.

3. Hyperbolicity cone. With every hyperbolic polynomial there is an associated open convex cone. Let p be a hyperbolic polynomial of degree m with respect to the direction d. The *hyperbolicity cone of* p *with respect to* d, written $C(p, d)$, is the set $\{x \in E : p(x + td) \neq 0,\ \forall t \geq 0\}$. In other words

$$C(p, d) = \{x \in E : t_i(x) > 0,\ 1 \leq i \leq m\}.$$

Since $t_i(rx) = rt_i(x)$ for all real $r > 0$, it is easy to see that $C(p, d)$ is indeed an open cone (that is, closed under positive scalar multiplication) with closure

$$\operatorname{cl} C(p, d) = \{x \in E : t_i(x) \geq 0,\ 1 \leq i \leq m\}.$$

The fact that it is also convex is deeper and we refer the reader to [4, Sect. 2] or to the more recent paper [5, Theorem 3.1]. Another important fact that can be found there is that if $c \in C(p, d)$ then p is also hyperbolic with respect to c and $C(p, d) = C(p, c)$. From now on the hyperbolicity cone will be denoted $C(p)$. We now return to the examples in the previous subsection and identify the hyperbolicity cone in each case.

(a) The hyperbolicity cone is the interior of the positive orthant:

$$\left\{x \in \mathbb{R}^n : x_i > 0,\ 1 \le i \le n\right\}.$$

(b) The hyperbolicity cone is the Lorenz cone:

$$\left\{x \in \mathbb{R}^n : \sqrt{x_2^2 + \cdots x_n^2} < x_1\right\}$$

(c) The hyperbolicity cone is the cone, S^n_{++}, of $n \times n$ symmetric positive definite matrices.

(d) The hyperbolicity cone is the interior of the operator norm epigraph

$$\{(X, r) \in M_{p,q} \times \mathbb{R} : |\sigma_i(X)| < r,\ 1 \le i \le q\},$$

where $\sigma_1(X)$, ..., $\sigma_q(X)$ are the singular values of the matrix X [3, Sect. 7.3].

2. Main result

In this section we prove our main result, first derived in [7][1]. We begin with a lemma.

Lemma 1. *For any real numbers t_1, ..., t_m, the following inequality holds:*

$$\left|\sum_{i=1}^m t_i^3\right| \le \left(\sum_{i=1}^m t_i^2\right)^{3/2}.$$

The next theorem is our key result.

Theorem 1. *Let p be a hyperbolic polynomial (homogeneous of degree m) with hyperbolicity cone $C(p)$. Let $a \ge 0$ be a real number and*

$$C_{>a}(p) = \{x \in C(p) : p(x) > a\}.$$

Then the function

$$f(x) = -m \log(p(x) - a)$$

is an m^2−self-concordant barrier on the set $C_{>a}(p)$.

Proof. The case $a = 0$ was proved in [5].

Step 0. For $x \in C_{>a}(p)$ and $h \in \mathbb{R}^n$, we can write, using the definition of $t_i(h, x)$,

$$p(x + th) = t^m p\left(h + \frac{1}{t}x\right) = t^m p(x) \prod_{i=1}^m \left(\frac{1}{t} + t_i(h, x)\right) = p(x) \prod_{i=1}^m (1 + tt_i).$$

What is important is that the roots $t_i = t_i(h, x)$ don't depend on the variable t. Differentiating both sides of the above representation we get the directional derivative of $p(x)$ in the direction of h, which is used below repeatedly:

$$\frac{d}{dt} p(x + th) = p(x + th) \sum_{i=1}^m \frac{t_i}{1 + tt_i}.$$

[1] A. Nemirovski has pointed out that analogous results can be derived using the general theory developed in [8, Sect. 5.1].

Step 1. Observe that in the case $a \neq 0$ we only need to prove self-concordance for $a = 1$, because we can make the linear substitution $x = a^{1/m} y$ to obtain

$$f(a^{1/m} y) = -m \log(p(y) - 1) - m \log(a).$$

(See for example [8, p. 148].)

We now compute the directional derivatives of f along the direction h, using the representation from above

$$f(x + th) = -m \log \left(p(x) \prod_{i=1}^{m} (1 + tt_i) - 1 \right).$$

For short we introduce the notation

$$\alpha = p(x) - 1, \quad C_1 = \sum_{i=1}^{m} t_i, \quad C_2 = \sum_{i=1}^{m} t_i^2, \quad C_3 = \sum_{i=1}^{m} t_i^3, \tag{5}$$

and observe that in our situation $\alpha > 0$. Elementary calculation shows

$$\begin{aligned} Df(x)[h] &= -\frac{m(\alpha+1)}{\alpha} C_1, \\ D^2 f(x)[h,h] &= \frac{m(\alpha+1)}{\alpha^2} C_1^2 + \frac{m(\alpha+1)}{\alpha} C_2, \text{ and} \\ D^3 f(x)[h,h,h] &= -\frac{m(\alpha+1)(\alpha+2)}{\alpha^3} C_1^3 - \frac{3m(\alpha+1)}{\alpha^2} C_1 C_2 - \frac{2m(\alpha+1)}{\alpha} C_3. \end{aligned}$$

We want to prove that inequalities (1) and (3) hold for every $h \in \mathbb{R}^n$ and $x \in C_{>1}(p)$.

Step 2. We start with inequality (3), which in the new notation is

$$\left| \frac{m(\alpha+1)}{\alpha} C_1 \right| \leq m \left(\frac{m(\alpha+1)}{\alpha^2} C_1^2 + \frac{m(\alpha+1)}{\alpha} C_2 \right)^{1/2}.$$

Notice that we can assume $\alpha + 1 \geq m$: the other case is trivial. After squaring both sides and dividing by $\frac{m^2(\alpha+1)}{\alpha}$ we get

$$\frac{(\alpha+1)}{\alpha} C_1^2 \leq \frac{m}{\alpha} C_1^2 + m C_2,$$

so we want to show

$$\frac{\alpha+1-m}{\alpha} C_1^2 \leq m C_2.$$

The Cauchy-Schwarz inequality gives us $C_1^2 \leq m C_2$ so since $m \geq 1$ we obtain

$$\frac{\alpha+1-m}{\alpha} C_1^2 \leq m \frac{\alpha+1-m}{\alpha} C_2 \leq m C_2,$$

as required.

Step 3. Now we turn our attention to inequality (1). With the new notation, this is

$$m\left|\frac{(\alpha+1)(\alpha+2)C_1^3}{\alpha^3}+\frac{3(\alpha+1)C_1C_2}{\alpha^2}+\frac{2(\alpha+1)C_3}{\alpha}\right| \\ \leq 2\left(\frac{m(\alpha+1)}{\alpha^2}C_1^2+\frac{m(\alpha+1)}{\alpha}C_2\right)^{3/2}.$$

We multiply both sides by $\frac{\alpha^3}{m(\alpha+1)}$ to get

$$\left|(\alpha+2)C_1^3+3\alpha C_1C_2+2\alpha^2C_3\right| \leq 2\sqrt{m(\alpha+1)}\left(C_1^2+\alpha C_2\right)^{3/2}.$$

Since this inequality is homogeneous in the vector $(t_1, t_2, ..., t_m)$, we may assume without loss of generality that $C_1 = \pm 1$. Furthermore, by multiplying this vector by -1 if necessary, we can suppose $C_1 = +1$, so the inequality becomes

$$\left|2+\alpha+3\alpha C_2+2\alpha^2C_3\right| \leq 2\sqrt{m\alpha+m}\ (1+\alpha C_2)^{3/2}.$$

We now square both sides and expand:

$$4 + \alpha^2 + 9\alpha^2C_2^2 + 4\alpha^4C_3^2 + 4\alpha + 12\alpha C_2 + 8\alpha^2C_3 + 6\alpha^2C_2 + 4\alpha^3C_3 + 12\alpha^3C_2C_3 \leq 4m\alpha + 12m\alpha^2C_2 + 12m\alpha^3C_2^2 + 4m\alpha^4C_2^3 + 4m + 12m\alpha C_2 + 12m\alpha^2C_2^2 + 4m\alpha^3C_2^3.$$

Regrouping gives

$$\begin{aligned} 0 \leq{}& \big(4mC_2^3-4C_3^2\big)\alpha^4+\big(4mC_2^3+12mC_2^2-4C_3-12C_2C_3\big)\alpha^3 \\ &+\big(12mC_2^2+12mC_2-6C_2-8C_3-9C_2^2-1\big)\alpha^2 \\ &+\big(12mC_2+4m-12C_2-4\big)\alpha+(4m-4). \end{aligned} \tag{6}$$

We show now that all the coefficients are positive. Using Lemma 1 and the fact $m \geq 1$, $C_2 \geq \frac{1}{m}$ this becomes clear for the coefficients of α^4, α and the constant term. Further, for the coefficient of α^3 using Lemma 1 we have

$$\begin{aligned} 4mC_2^3+12mC_2^2-4C_3-12C_2C_3 &\geq 4mC_2^3+12mC_2^2-4C_2^{3/2}-12C_2^{5/2} \\ &= C_2^{3/2}\big(4mC_2^{3/2}+12mC_2^{1/2}-4-12C_2\big). \end{aligned}$$

Consider the polynomial $q(s) := 4ms^3-12s^2+12ms-4$. Its derivative $q'(s) = 12(ms^2-2s+m)$ is nonnegative, so q is increasing. Using the fact that $\frac{1}{\sqrt{m}} \leq C_2^{1/2}$ we get

$$\begin{aligned} q(C_2^{1/2}) \geq q\left(\frac{1}{\sqrt{m}}\right) &= \frac{4\sqrt{m}}{m}-\frac{12}{m}+\frac{12m\sqrt{m}}{m}-\frac{4m}{m} \\ &= \frac{4(\sqrt{m}-1)+8(m\sqrt{m}-1)+4m(\sqrt{m}-1)}{m} \geq 0, \end{aligned}$$

which shows that the coefficient of α^3 is positive. For the coefficient of α^2, using Lemma 1, we have

$$\begin{aligned}
&12mC_2^2 + 12mC_2 - 6C_2 - 8C_3 - 9C_2^2 - 1\\
&\quad \geq 12mC_2^2 + 12mC_2 - 6C_2 - 8C_2^{3/2} - 9C_2^2 - 1\\
&\quad = 9(m-1)C_2^2 + 6(m-1)C_2 + (mC_2 - 1) + C_2\big(3mC_2 - 8C_2^{1/2} + 5m\big).
\end{aligned}$$

The quadratic polynomial $3ms^2 - 8s + 5m$ is strictly positive in the case when $m \geq 2$, and using the fact that $C_2 \geq \frac{1}{m}$ this implies that the coefficient is positive. In the case when $m = 1$ we have $C_2 = 1$ and one immediately sees that the coefficient of α^2 is actually zero. The fact that all coefficients of the quadric polynomial on the right hand side of inequality (6) are positive implies that the inequality holds for all $\alpha \geq 0$, which is what we wanted to prove.

□

3. Examples and applications

3.1. Examples

Following our examples from Sect. 1, we obtain the following applications of the main result.

(a) For any natural number m the function

$$f(x_1, \ldots, x_m) = -m \log\left(\prod_{i=1}^{m} x_i - 1\right)$$

is an m^2-self-concordant barrier on the set

$$\left\{x \in \mathbb{R}^m : \prod_{i=1}^{m} x_i > 1,\ x_i > 0,\ 1 \leq i \leq m\right\}.$$

In particular when $m = 2$ this result follows from Proposition 5.3.2 in [8].

(b) The function

$$f(x, y) = -2\log(y^2 - \|x\|^2 - 1)$$

is a 4-self-concordant barrier on the set

$$\left\{(y, x) \in \mathbb{R} \times \mathbb{R}^{n-1} : y \geq \sqrt{\|x\|^2 + 1}\right\}.$$

This result can also be found in [8]. (See the paragraph following the proof of Proposition 5.4.3 and make the linear substitution $t \to z - 1$, $y \to z + 1$ in the function Ψ.)

(c) A more interesting example is the function

$$f(X) = -m\log(\det X - 1),$$

which is an m^2-self-concordant barrier on the set

$$\{X \in S^m_{++} : \det X > 1\}.$$

(d) The function

$$f(X, r) = -2q\log\left(\det\left(X^T X - r^2 I_q\right) - 1\right)$$

is a $(2q)^2$-self-concordant barrier on the set

$$\left\{(X, r) \in M_{p,q} \times \mathbb{R} : \det\left(X^T X - r^2 I_q\right) > 1\right\}.$$

3.2. Application: hyperbolic means

A *hyperbolic mean* is a function of the form $p(x)^{1/m}$, where p is a hyperbolic polynomial of degree m, and the domain is the hyperbolicity cone $C(p)$. Hyperbolic means are positively homogeneous and concave [5, Lemma 3.1]. Examples include the geometric mean $(\prod_{i=1}^m x_i)^{1/m}$, and the function

$$X \in S^m_{++} \mapsto (\det X)^{1/m}.$$

A natural approach to applying interior point methods to convex programs involving hyperbolic means is to use a self-concordant barrier for the *hypograph* of the mean, the convex cone

$$H(p) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in C(p),\ 0 < t^m < p(x)\}.$$

The following result provides such a barrier.

Theorem 2. For a suitable positive real μ (for example $\mu = 400$), if p is a hyperbolic polynomial of degree m then

$$(x, t) \mapsto -\mu m\left(\log\left(\frac{p(x)}{t^m} - 1\right) + 2m\log t\right)$$

is a $2\mu m^2$-normal barrier for the hypograph, H(p), of the hyperbolic mean.

Proof. Apply Proposition 5.1.4 in [8] to Theorem 1. □

As a simple-minded illustration, suppose we want to solve the problem

$$\begin{array}{ll} \sup & p(x)^{\frac{1}{m}} + \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in C(p), \end{array}$$

for some linear map A and given b and c. Rewrite this problem in the equivalent form

$$\begin{array}{ll} \sup & t + \langle c, x\rangle \\ \text{s.t.} & t < p(x)^{\frac{1}{m}} \\ & Ax = b \\ & x \in C(p), \end{array}$$

and finally into the form

$$\begin{array}{ll} \max & \langle \tilde{c}, \tilde{x}\rangle \\ \text{s.t.} & \tilde{A}\tilde{x} = b \\ & \tilde{x} \in H(p), \end{array}$$

where $\tilde{c} := (c, 1)$, $\tilde{x} := (x, t)$, $\tilde{A}(x, t) := Ax$. We have an easily computable self-concordant (logarithmically homogeneous) barrier for the cone $H(p)$, so we can design an interior point algorithm to solve this hyperbolic mean maximization problem. Using this result we can as well easily model convex programs with constraints involving hyperbolic means, since $x \in C(p)$ satisfies an inequality of the form

$$\langle c, x\rangle - p(x)^{1/m} < b$$

if and only if there exists positive real t satisfying

$$\langle c, x\rangle - t < b, \quad t^m < p(x).$$

In [8, p. 239], Nesterov and Nemirovskii show how to model convex programs involving the geometric mean or $(\det(\cdot))^{1/m}$ by semidefinite programming. It is interesting to compare their approach to this idea. Their approach involves additional variables ($O(m^2)$ variables to model $\det(\cdot)^{1/m}$, for example), whereas this idea is direct and applies to any hyperbolic mean. On the other hand, extremely efficient algorithms are now available for semidefinite programming (see for example [1], [9]).

4. Relationship with Güler's result

As we mentioned above, in [5] Güler proved that $-\log(q(x))$ is an n-self-concordant barrier on $C(q)$ for any hyperbolic polynomial q of degree n. (Güler attributes the observation to Renegar.) In this concluding section we want to show that our result cannot be deduced as an "elementary" consequence of this fact, by which we mean that we cannot take a self-concordant barrier of the above type, restrict it to an affine subspace and obtain the self-concordance of $-m\log(p(x) - 1)$.

Consider the following special case of Theorem 1:

$$-3\log(x^3 - 1) \text{ is self-concordant on } (1, +\infty).$$

To deduce this from [5] we would need a hyperbolic polynomial q with respect to d with hyperbolicity cone $C(q)$ and vectors a and b such that

$$\begin{aligned} &(x^3 - 1)^3 = q(a + xb), \text{ for all } x \in \mathbb{R}, \text{ and} \\ &1 < x \in \mathbb{R} \Leftrightarrow a + xb \in C(q). \end{aligned}$$

When $x = 0$ we immediately get $q(a) = -1$. We can also conclude that $b \in \operatorname{cl} C(q)$ - a closed convex cone. Since $d \in C(q)$, an open convex cone, we have for all small enough real $\epsilon > 0$, that $b + \epsilon d \in C(q)$, so the polynomial q is hyperbolic with respect to $b + \epsilon d$ as well. That is, for all small enough $\epsilon > 0$ the polynomial (in x) $q(a + x(b + \epsilon d))$ has only real, nonzero roots. Clearly if $q(a + xb) = (x^3 - 1)^3$ then $n \geq 9$. We divide both sides of this equality by x^n, and setting $t := 1/x$ obtain

$$q(at + b) = t^{n-9} - 3t^{n-6} + 3t^{n-3} - t^n = t^{n-9}(1 - t^3)^3.$$

Using the fact that $q(a + x(b + \epsilon d))$ has nonzero roots and applying the same substitution as above we get that the polynomial (in t) $t \mapsto q(at + b + \epsilon d)$ has only real roots. Now, for ϵ close to zero, the degree of the polynomial $q(at + b + \epsilon d)$ is constant, and so its roots approach the roots of $q(at + b)$ as ϵ approaches zero. This is a contradiction with the fact that $q(at + b)$ has a complex root.

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