Efficient Estimation of Liquidity-Adjusted Risk Measures

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27. August 2009
Deutsche Bundesbank, Frankfurt am Main

The views herein are my own and do not necessarily reflect those of Deutsche Bundesbank.
Goals:

1. Introduce a framework that incorporates market liquidity risk into portfolio models.
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- Construct an estimation procedure for liquidity-adjusted risk measures.
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Outline:

- Liquidity-adjustment to portfolio models
- Liquidity-adjusted convex risk measures
- Stochastic root finding methods
Motivation: Liquidity Risk

Idea:

- Quantify risk of individual portfolio.
- Determine an minimal cash position, such that the portfolio is acceptable.
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Liquidity Risk:

- Price of an asset is determined by a supply-and-demand curve.
- Trading of large positions moves the price.
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Liquidity Risk:

- Price of an asset is determined by a supply-and-demand curve.
- Trading of large positions moves the price.
- Portfolio managers can be forced to liquidate:
  - Cashflows
  - Construction of portfolio
Example: Investment Financed with Bond

- An investor issues bond to finance an investment (risky asset) at time $t = 0$.
- Bond requires coupon payment at time $t = 1$.
- Expected return of risky asset is higher than the coupon.
- At time $t = 1$ the investor liquidates part of the asset to pay the coupon.
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- At time $t = 1$ the investor liquidates part of the asset to pay the coupon.
- **Market liquidity**: Liquidation lowers price of the asset (hence value of the portfolio).
Example: Long-Short Investment Strategy

- A fund manager invests according to a long-short strategy at time $t = 0$.
- The manager has a constraint on the overall short position.
- If price of the short position rises, he will be forced to liquidate part of the short and possibly long position to meet the portfolio constraint.
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• A fund manager invests according to a long-short strategy at time $t = 0$.

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• If price of the short position rises, he will be forced to liquidate part of the short and possibly long position to meet the portfolio constraint.

• **Market liquidity**: Liquidation lowers value of the portfolio significantly.

**Question:** How big does the cash position need to be, such that the portfolio is acceptable?
Acceptable Positions

Random variable $L \in L^\infty$ represents loss of a portfolio.

1. Construct a loss distribution.
2. Define and evaluate appropriate risk measure $\rho : L^\infty \to \mathbb{R}$.
3. A position with loss variable $L$ is called acceptable, if

$$\rho(L) \leq 0.$$  

$\rho$ induces an acceptance set

$$\mathcal{A} := \{L \in L^\infty | \rho(L) \leq 0\}.$$
Liquidity-Adjustment
Portfolio Set-Up

- Consider a Portfolio of one cash-position ($\xi_0$) and $N$ risky assets:

$$\xi = (\xi_0, \xi_1, \ldots, \xi_N) \in \mathbb{R}^{N+1}.$$ 

- An investor wants to hold portfolio $\xi$ for one time-period (from $t = 0$ to $t = 1$).

- **Mark-to-Market** Value of the portfolio at time $t$:

$$V^{(t)}(\xi) := \xi_0 + \sum_{i=1}^{N} m_i^{(t)} \cdot \xi_i,$$

  $m_i^{(t)}$ price of $i$th asset at time $t$.

- Loss at time $t = 1$

$$L(\xi)(\omega) := -(V^{(1)}(\xi)(\omega) - V^{(0)}(\xi)).$$
Portfolio Model with Liquidity-Adjustment

Model:

- **Cashflows:** \( \phi : \mathbb{R}^{N+1} \rightarrow \mathbb{R}, \xi \mapsto \phi(\xi) \) at time \( t = 1 \).
- **Liquidity constraint:** cash position needs to be non-negative.
- **Portfolio constraint:** \( \mathcal{H} \subset \mathbb{R}^N \), e.g.

\[
\mathcal{H} = [-q_i, \infty)^N, \quad i = 1, \ldots, N.
\]

- **Trading moves price:** Marginal supply-demand curves at time \( t = 1 \):

\[
\tilde{m} = (\tilde{m}_1^{(1)}, \ldots, \tilde{m}_N^{(1)}),
\]

where \( \tilde{m}_i^{(1)} : \mathbb{R} \rightarrow \mathbb{R} \) is a decreasing function. For example:

\[
\tilde{m}_i^{(1)}(x) = m_i^{(1)} - b \cdot x, \quad b \geq 0,
\]

where \( x \) is the number of liquidated shares.

In this setting mark-to-market valuation at time \( t = 1 \) is too optimistic.
Introduction

Liquidity-Adjustment

Convex Risk Measures

Stochastic Root Finding

Simulation

References

Attainable and Liquid Portfolios at Time \( t = 1 \)

**Definition:** For fixed portfolio \( \xi \in \mathbb{R}^{N+1} \), the portfolio \( \zeta \in \mathbb{R}^{N+1} \) is attainable \( (\zeta \in \mathcal{R}(\xi, m^{(1)})) \) if there exists \( \kappa \in \mathbb{R}^N \) such that

\[
\zeta = \left( \xi_0 + \sum_{i=1}^{N} \int_{0}^{\kappa_i} m_i^{(1)}(x) \, dx, \xi - \kappa \right).
\]
Attainable and Liquid Portfolios at Time $t = 1$

**Definition:** For fixed portfolio $\xi \in \mathbb{R}^{N+1}$, the portfolio $\zeta \in \mathbb{R}^{N+1}$ is attainable ($\zeta \in \mathcal{R}(\xi, m^{(1)})$) if there exists $\kappa \in \mathbb{R}^N$ such that

$$
\zeta = \left( \xi_0 + \sum_{i=1}^{N} \int_0^{\kappa_i} m^{(1)}_i(x) \, dx, \xi - \kappa \right).
$$

**Definition:** The set of liquid portfolios at time $t = 1$ are

$$
\mathcal{L}(\xi, m^{(1)}, \phi) = \left\{ \zeta \in \mathcal{R}(\xi, m^{(1)}): \xi_0 + \phi(\xi) \geq 0 \right\}.
$$
Definition: Value of portfolio $\xi$ at time $t = 1$

$$V(\xi, m^{(1)}, \phi, \mathcal{H}) = \sup \left\{ \sum_{i=0}^{N} \zeta_i \cdot \tilde{m}_i^{(1)} : \zeta \in \mathcal{L}(\xi, m^{(1)}, \phi) \cap \mathcal{H} \right\}.$$ 

The valuation process is an optimization problem under liquidity and portfolio constraints.
Value of Portfolio at Time $t=1$

Definition: Value of portfolio $\xi$ at time $t = 1$

$$V(\xi, m^{(1)}, \phi, \mathcal{H}) = \sup \left\{ \sum_{i=0}^{N} \zeta_i \cdot \tilde{m}_i^{(1)} : \zeta \in \mathcal{L}(\xi, m^{(1)}, \phi) \cap \mathcal{H} \right\}.$$

The valuation process is an optimization problem under liquidity and portfolio constraints.

Definition: Random value of portfolio $\xi$ at time $t = 1$

$$\Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$$

$$\omega \longmapsto V(\xi, m^{(1)}(\omega), \phi(\omega), \mathcal{H}).$$
Convention:

- $V^{(0)}(\xi) = 0$, hence $L = -V^{(1)}(\xi) = -V(\xi)$.
- $\xi + k := (\xi_0 + k, \xi_1, \ldots, \xi_N)$ for $k \in \mathbb{R}$.

Definition: For given risk measure $\rho$ with acceptance set

$$\mathcal{A} = \{L \in L^\infty | \rho(L) \leq 0\},$$

the liquidity-adjusted risk of a portfolio $\xi$ is defined as

$$\rho^V(\xi) := \inf \{k : -V(\xi + k) \in \mathcal{A}\}.$$

The model is based on work of Acerbi et al. (2008) and Anderson et al. (2009).
Convex Risk Measures
The Industrial Standard Value-at-Risk

For a given confidence level $\alpha \in (0, 1)$:

$$\text{VaR}_\alpha(L) = F_L^{-1}(\alpha).$$
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VaR has significant drawbacks:

- Does not account for the size of extremely large losses
- Does not encourage diversification
- Provides incentives to take riskier positions
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What are criteria for a good risk measure?

This question motivated an axiomatic analysis of risk measures (initiated by Artzner, Delbaen, Eber and Heath (1999)).
A risk measure $\rho : L^\infty \to \mathbb{R}$ is convex if:

- **Monotonicity**: If $L_1 \leq L_2$, then $\rho(L_1) \leq \rho(L_2)$
- **Cash invariance**: If $m \in \mathbb{R}$, then $\rho(L - m) = \rho(L) - m$
- **Convexity** (Föllmer and Schied (2002)):

  $$\rho(\alpha L_1 + (1 - \alpha)L_2) \leq \alpha \rho(L_1) + (1 - \alpha)\rho(L_2), \quad \alpha \in [0, 1].$$

A subclass called **coherent** risk measures additionally satisfy

- **Positive homogeneity**: If $\lambda \geq 0$, then $\rho(\lambda L) = \lambda \rho(L)$. 

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Axiomatic Approach: Convex Risk Measures
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- **Positive homogeneity**: If $\lambda \geq 0$, then $\rho(\lambda L) = \lambda \rho(L)$.

Industrial standard Value at Risk is **NOT** convex!
A Class of Convex Risk Measures

$I : \mathbb{R} \rightarrow \mathbb{R}$ convex, increasing, non-constant function (loss function), $x_0$ interior point of the range of $I$.

Utility-Based Shortfall Risk (UBSR) (Föllmer and Schied (2002)):

$$\rho_{UBSR}(L) = \inf \{ m \in \mathbb{R} : E [I(L - m)] \leq x_0 \}.$$
UBSR: $\rho_{UBSR}(L)$ corresponds to the unique root $s^*$ of the function

$$\tilde{g}(s) := \mathbb{E}[l(L - s)] - x_0,$$

$\tilde{g}$ is decreasing and convex.

Dunkel and Weber (2009) use stochastic root finding to estimate $\rho_{UBSR}$. 

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Liquidity-Adjusted Risk Measures revisited

Convention: \( V^{(0)}(\xi) = 0 \), i.e. \( L = -V^{(1)}(\xi) = -V(\xi) \).

\( \rho : L^\infty \to \mathbb{R} \) a risk measure with acceptance set

\[ \mathcal{A} = \{ -V(\xi) \in L^\infty : \rho(-V(\xi)) \leq 0 \} . \]

Liquidity-Adjusted risk of portfolio \( \xi \):

\[ \rho^V(\xi) := \inf \{ k : -V(\xi + k) \in \mathcal{A} \} . \]

Proposition: \( \rho \) is a convex risk measure and \( \rho \) and \( V \) are continuous, then

\[ \rho^V(\xi) = k^* \iff \rho(-V(\xi + k^*)) = 0. \]
Liquidity-Adjusted UBSR

Need to find \( k^* \) such that

\[
\rho_{UBSR}(\ -V(\xi + k^*)) = 0.
\]

By continuity,

\[
\rho_{UBSR}(\ -V(\xi + k)) = y \iff \tilde{g}(y) = \mathbb{E}[l(-V(\xi + k) - y)] - x_0 = 0,
\]

it follows that \( k^* \) is the unique root of

\[
g(k) := \mathbb{E}[l(-V(\xi + k)) - x_0].
\]

Stochastic root finding methods can be used to estimate the liquidity-adjusted UBSR risk measure.
Stochastic Root Finding
Stochastic Root Finding Methods

- Robbins-Monro
- Polyak-Ruppert
- Stochastic Averaging
Let \((\omega_n)_n\) be an i.i.d. sequence of scenarios.

Choose constants \(\gamma \in \left(\frac{1}{2}, 1\right], c > 0\), and a starting value \(k_1 \in [a, b]\) \((k^* \in [a, b])\).

For \(n \in \mathbb{N}\) define recursively:

\[
k_{n+1} = \Pi_{[a,b]} \left[ k_n + \frac{c}{n^\gamma} \cdot (l(\xi + k_n)(\omega_n) - x_0) \right].
\]  

(1)
Robbins-Monro: Theoretical Justification

**Theorem:** $V \in L^\infty, g \in C^1, c > (-2g'(k^*))^{-1}$.

If $\gamma = 1$, then

$$\sqrt{n} \cdot (k_n - k^*) \rightarrow N \left( 0, \frac{-c^2 \sigma^2(k^*)}{2cg'(k^*) + 1} \right). \quad (2)$$

If $\gamma \in \left( \frac{1}{2}, 1 \right)$, then

$$\sqrt{m^\gamma} \cdot (k_n - k^*) \rightarrow N \left( 0, \frac{-c\sigma^2(k^*)}{2g'(k^*)} \right). \quad (3)$$
Variations

• Polyak-Ruppert, for arbitrary $\rho \in (0, 1)$, $(k_i)_i$ sequence of Robbins-Monro

$$\bar{k}_n = \frac{1}{\rho \cdot n} \sum_{i=(1-\rho)n}^{n} k_i.$$

• Use Stochastic Averaging with Stochastic Approximation, define

$$T_n = \frac{1}{N} \sum_{i=1}^{N} (l(-V(\xi + k_n(\omega_i)) - x_0),$$

use $T_n$ for the update-step

$$k_{n+1} = \Pi(k_n + \frac{c}{n^\gamma} \cdot T_n).$$

• Importance sampling
Optimized Certainty Equivalents (OCE) (Ben-Tal and Teboulle (2007)):

\[ \rho_{OCE}(L) := \inf_{\eta \in \mathbb{E}} (\eta + \mathbb{E}[l(L - \eta)]) \]

is a \textit{convex} risk measure.
Optimized Certainty Equivalents (OCE) (Ben-Tal and Teboulle (2007)):

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\]

is a convex risk measure.

Remark: For \( l(y) = \frac{1}{\alpha} y \mathbb{1}_{\{y \geq 0\}}(y) \):

\[ 
\rho_{OCE}(L) = CVAR_\alpha(L). 
\]
Extension to Other Classes of Risk Measures

Optimized Certainty Equivalents (OCE) (Ben-Tal and Teboulle (2007)):

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Remark: For $$l(y) = \frac{1}{\alpha} y \mathbb{I}_{\{y \geq 0\}}(y)$$:

$$\rho_{OCE}(L) = CVAR_\alpha(L).$$

Representation: The following are equivalent

1. $$\rho_{OCE}(L) = \eta^* + \mathbb{E}[l(L - \eta^*)];$$
2. $$\mathbb{E}[l'(L - \eta^*)] = 1.$$

Need to find the root $$\eta^*$$ of

$$g(\eta) = \mathbb{E}[l'(L - \eta)] - 1.$$
Liquidity-Adjusted OCE Risk Measures

Representations:

\[ k^* = \rho_{OCE}(\xi) \iff \rho_{OCE}(-V(\xi + k^*)) = 0 \]

\[ \rho_{OCE}(-V(\xi + k^*)) = \mathbb{E}[l(-V(\xi + k^*) - \eta^*)] + \eta^* = 0 \]

\[ \mathbb{E}[l'(-V(\xi + k^*) - \eta^*) - 1] = 0 \]
Liquidity-Adjusted OCE Risk Measures

Representations:

\[ k^* = \rho_{OCE}(\xi) \iff \rho_{OCE}(-V(\xi + k^*)) = 0 \]

\[ \rho_{OCE}(-V(\xi + k^*)) = \mathbb{E}[l(-V(\xi + k^*) - \eta^*)] + \eta^* = 0 \]

\[ \mathbb{E}[l'(-V(\xi + k^*) - \eta^*) - 1] = 0 \]

Need to find \( \theta^* = (k^*, \eta^*) \) such that \( g(\theta^*) = 0 \), for \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \),

\[ g(\theta) := \left( \begin{array}{c} \mathbb{E}[l(-V(\xi + k) - \eta) + \eta] \\ \mathbb{E}[l'(-V(\xi + k) - \eta) - 1] \end{array} \right). \]
Define random vector

\[ Y_{k,\eta} = \begin{pmatrix} l(-V(\xi + k) - \eta) + \eta \\ l'(-V(\xi + k) - \eta) - 1 \end{pmatrix}, \]

2-dimensional Robbins-Monro iteration:

\[ \theta_{n+1} = \Pi_H \left[ \theta_n + \frac{c}{n^\gamma} \cdot Y_{k_n,\eta_n} \right]. \]
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  l(-V(\xi + k) - \eta) + \eta \\
  l'(V(\xi + k) - \eta) - 1
\end{pmatrix},
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2-dimensional Robbins-Monro iteration:

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\theta_{n+1} = \Pi_H \left[ \theta_n + \frac{c}{n^{\gamma}} \cdot Y_{k_n,\eta_n} \right].
\]

**Theorem:** Assume \( g \) is Lipschitz-continuous and \( \sup_{\theta \in H} \mathbb{E}[|Y_\theta|^2] < \infty \), then

\[
\theta_n \longrightarrow \theta^* \text{ a.s.}
\]
Simulation Example
Example: Investment Financed with Bond rev.

- Investor issues bond \((m_1 = 10)\) to invest in profitable asset \((m_2^{(0)} = 10)\).
- At time \(t = 1\): liquidates part of assets to cover required coupon payment \(c = 1.5\).
- Trading in asset moves the price

\[
\tilde{m}_2^{(1)}(x) = m_2^{(1)} - b \cdot x, \quad m_2^{(1)} \sim LN(13, 0.3).
\]

- Investor needs to meet liquidity and portfolio constraints.
- Portfolio needs to be acceptable according risk measure \(\rho\).
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**Question:** How big does the cash position need to be, such that the portfolio becomes acceptable?
Answer:

Liquidity-adjusted risk measure $ES_{1\%}^V$ and $VaR_{1\%}^V$
Example: Long-Short Strategy rev.

- Portfolio manager short sells risky asset \(m_1\) and buys another risky asset \(m_2\).
- Short positions are constrained.
- At time \(t = 1\), the manager needs to liquidate part of the short and possibly long position to meet portfolio constraints.
- Trading at \(t = 1\) moves prices

\[
m_1(x) = m_1^{(1)} - b_1 \cdot x, \quad b_1 \geq 0,
\]
\[
m_2(x) = m_2^{(2)} - b_2 \cdot x, \quad b_2 \geq 0,
\]

where

\[
\begin{pmatrix}
  m_1^{(1)} \\
  m_2^{(2)}
\end{pmatrix}
\sim N\left(\begin{pmatrix}
  10 \\
  10.2
\end{pmatrix}, \begin{pmatrix}
  0.2 & 0.18 \\
  0.18 & 0.2
\end{pmatrix}\right).
\]
Example: Long-Short Strategy cont.

Question: How big does the cash position need to be, such that the portfolio becomes acceptable?

Assume OCE risk measure with loss function $l(y) = 20 \cdot y^2 \cdot \mathbb{1}_{\{y \geq 0\}}(y)$. 
Liquidity effect: OCE with quadratic loss

- OCE
- short position $b_1$
- long position $b_2$
Convergence of 2-Dimensional Robbins-Monro

- 50 paths of Robbins-Monro algorithm
- Mean of 50 paths of Robbins-Monro algorithm

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Efficient Estimation of Liquidity-Adjusted Risk Measures
Conclusions:

- Liquidity-adjusted risk measures can be used to determine the smallest necessary cash position of a portfolio under liquidation effects.
Conclusions:

- Liquidity-adjusted risk measures can be used to determine the smallest necessary cash position of a portfolio under liquidation effects.
- Stochastic root finding methods provide a helpful tool to estimate risk measures.
Further Research:

- Application in practice
- Dynamic set-up
- Thorough comparison of suggested root finding methods
- ...
THANK YOU!


