Recitation 7A: Basic problems

In this recitation, we will try to get a better understanding on how dynamic programming works and how we can formulate various problems as dynamic programming problems. You do not have to turn in this week's recitation. Instead, take this handout home with you for additional practice with dynamic programming.

Name and NetID:
Section:

1 The knapsack problem, version 1

(The knapsack problem) You have a knapsack with weight capacity $W = 13$ pounds. There are 4 different products: products 1, 2, 3, and 4. Each copy of product i has a value d_i dollars and weight w_i pounds, given below:

Product (i)	Weight per unit (w_i)	Value per unit (d_i)
1	5	9
2	3	4
3	2	3
4	1	0.5

There is an infinite number of copies of each product available. However, due to the weight capacity of your knapsack, you have to make a wise choice on how many copies of each product you choose to put in your knapsack in order to maximize the total value (of your knapsack content).

This problem should remind you of problem 2 in Homework 5. There are various ways in which you can formulate this problem. (The following method could be different from how you did it in the homework.)

1. Suppose we consider a simpler problem: For instance, if the capacity is $\hat{W} = 0$ instead of $W = 13$, then we can fit none of the products, so the maximum value of our knapsack content is 0.

   Hence, supposing that

   $$\boxed{f^*(\widehat{W}) \text{ denote the maximum value if the knapsack capacity is } \widehat{W},}$$

   then $f^*(0) =$ _______.

   Next, suppose that our knapsack capacity is only $\widehat{W} = 1$ instead of $W = 13$. We note that only product 4 can fit. Then, what is the corresponding maximum value that you could possibly have?

   So, $f^*(1) =$ _______.

2. Suppose, now, that the capacity of your knapsack is slightly larger, $\widehat{W} = 2$. Our question is: based on our observation in (a), which one copy of any of the products 1, 2, 3, and 4 will you choose to add into your knapsack? We guide you below:

   - After noting that products 1 and 2 are too heavy, we see that we can choose either one copy of product 3 or one copy of product 4.
   - If you choose to add one copy of product 3, what is the remaining weight capacity?

     What is the maximum value that we can achieve given this remaining weight?

     Max value if our knapsack has capacity $\widehat{W} = 2$ *and we choose product 3*
     $= \_\_\_\_\_\_\_ + f^*(\text{remaining weight}) = ...$

     Then, what is the corresponding maximum value that you could possibly have if you added one copy of product 3 here?

   - If you choose to add one copy of product 4, what is the remaining weight capacity?

     What is the maximum value that we can achieve given this remaining weight?

     Max value if our knapsack has capacity $\widehat{W} = 2$ *and we choose product 4*
     $= \_\_\_\_\_\_\_ + f^*(\text{remaining weight}) = ...$

     Then, what is the corresponding maximum value that you could possibly have if you added one copy of product 4 here?

   - Based on the previous two observations, would you choose product 3 or product 4, if your knapsack's capacity is $\widehat{W} = 2$?

     What is the corresponding maximum value that you can achieve if your knapsack has a capacity of $\widehat{W} = 2$?

     So, $f^*(2) = \max\{\_\_\_\_\_ + f^*(\_\_\_\_\_), \_\_\_\_\_ + f^*(\_\_\_\_\_)\} = \_\_\_\_\_$.

3. We reproduce the table of item weights and value as follows.

Product (i)	Weight per unit (w_i)	Value per unit (v_i)
1	5	9
2	3	4
3	2	3
4	1	0.5

Then, **continuing the method above, complete the following table**.
(Recall that $f^*(\widehat{W})$ denote the maximum value if the knapsack capacity is $\widehat{W}$.)

$\widehat{W}$	Max value if we choose one copy of				Max value	Achieved
	product 1	product 2	product 3	product 4	$(f^*(\widehat{W}))$	by prod:
0	-	-	-	-	0	none
1	-	-	-	$0.5 + f^*(0) = 0.5$	0.5	4
2	-	-	$3 + f^*(0) = 3$	$0.5 + f^*(1) = 1$	3	3
3	-					
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						

So, the maximal value if our knapsack has capacity $W = 13$ is

$$f^*(13) = \_\_\_\_\_\_,$$

and the optimal number of copies of each product to take in our knapsack is:

$$x_1 = \_\_\_\_\_\_, x_2 = \_\_\_\_\_\_, x_3 = \_\_\_\_\_\_, x_4 = \_\_\_\_\_\_.$$

Note. To find the values of x_1, x_2, x_3, x_4, consider the following:

- According to our table, when our knapsack capacity is 13, which product did we choose to add? Adding this product, what is our remaining knapsack capacity?

- Suppose that our remaining knapsack capacity (from the above step) is $\bar{W}$. According to our table, when our knapsack capacity is $\bar{W}$, which product did we choose to add? Adding this product, what is our remaining knapsack capacity?

- Continue asking the above questions until we cannot add any more copy of any product into our knapsack. Count how many times we add each of the four products? This will give you the optimal number of products to be added into the knapsack.

Formalizing the DP approach above:

- Note that we solve for $f^*(\widehat{W})$ starting from $\widehat{W} = 0$ ending at $\widehat{W} = 13$, increasing by 1 unit each time. This illustrate how we solve for $f^*(\widehat{W})$ in **stages**, starting from the smallest value of $\widehat{W}$, namely 0 to $\widehat{W} = 13$.
  So, stage $\widehat{W}$ corresponds to remaining weight capacity of $\widehat{W}$.
  For example, when we want to solve for $f^*(10)$, we will need to know what the values of $f^*(9), f^*(8), f^*(7), f^*(5)$ first. So, we have to solve for $f^*(\widehat{W})$ from smallest to larger values of $\widehat{W}$.
- In this case, at each stage $f^*(\widehat{W})$. So, at stage $\widehat{W}$, we consider one possible state only: $\widehat{W}$.
- Given the above stages and states information, describe in words the decisions that we consider at each state in each stage.
- What is the word-description of $f^*(\widehat{W})$?
- Which value of $\widehat{W}$ did we compute $f^*(\widehat{W})$ first? This is our "boundary condition".
- Give a recurrence relation for $f^*(\widehat{W})$ based on our method above.
- For what value of $\widehat{W}$ does $f^*(\widehat{W})$ corresponds to the maximum value of our original knapsack problem?

2 Knapsack problem, approach # 2

We consider the exact same knapsack problem, but using a different DP approach. Recall that we have a knapsack of weight capacity $W = 13$ pounds and 4 items:

Product (i)	Weight per unit (w_i)	Value per unit (d_i)
1	5	9
2	3	4
3	2	3
4	1	0.5

1. Suppose we consider a simpler problem: Suppose that we only have one product to choose from, namely product 1, then this problem is easy. Suppose x_1^* denote the optimal number of copies of product 1 to take, assuming that the weight capacity leftover for product 1 is $\widehat{W}$: For any leftover weight capacity $\widehat{W} \in \{0, 1, \ldots, 13\}$, we take as many copies of product 1 as we can fit into our knapsack:

$$x_1^* = \left\lfloor \frac{\widehat{W}}{w_1} \right\rfloor = \left\lfloor \frac{\widehat{W}}{5} \right\rfloor$$

Hence, supposing that

$\boxed{f_1^*(\widehat{W}) \text{ denote the maximum value if we only have product 1 and the knapsack capacity is } \widehat{W},}$

then

$$\begin{aligned}
f_1^*(0) &= 9 * x_1 = 0, \quad x_1^* = 0 \\
f_1^*(1) &= 9 * x_1 = 0, \quad x_1^* = 0 \\
f_1^*(2) &= 9 * x_1 = 0, \quad x_1^* = 0 \\
f_1^*(3) &= 9 * x_1 = 0, \quad x_1^* = 0 \\
f_1^*(4) &= 9 * x_1 = 0, \quad x_1^* = 0 \\
f_1^*(5) &= 9 * x_1 = 9, \quad x_1^* = 1 \\
f_1^*(6) &= 9 * x_1 = 9, \quad x_1^* = 1 \\
f_1^*(7) &= 9 * x_1 = 9, \quad x_1^* = 1 \\
f_1^*(8) &= 9 * x_1 = 9, \quad x_1^* = 1 \\
f_1^*(9) &= 9 * x_1 = 9, \quad x_1^* = 1 \\
f_1^*(10) &= 9 * x_1 = 18, \quad x_1^* = 2 \\
f_1^*(11) &= 9 * x_1 = 18, \quad x_1^* = 2 \\
f_1^*(12) &= 9 * x_1 = 18, \quad x_1^* = 2 \\
f_1^*(13) &= 9 * x_1 = 18, \quad x_1^* = 2
\end{aligned}$$

2. Next, suppose that we have two products: product 1 and product 2. This problem is more complicated since we have to consider all of the possible combinations of quantities of product 1 and product 2 that can fit, which could be a lot.

However, suppose we focus on the possible quantities of product 2 that we can take, given a weight capacity of $\widehat{W}$:

$\widehat{W}$	Max value, if $x_2 = 0$	Max value if $x_2 = 1$	Max value if $x_2 = 2$	Max value if $x_2 = 3$	Max value if $x_2 = 4$	$f_2^*(\widehat{W})$	x_2^*
0	$0 \cdot 4 + f_1^*(0) = 0$	-	-	-	-	0	0
1	$0 \cdot 4 + f_1^*(1) = 0$	-	-	-	-	0	0
2	$0 \cdot 4 + f_1^*(2) = 0$	-	-	-	-	0	0
3	$0 \cdot 4 + f_1^*(3) = 0$	$1 \cdot 4 + f_1^*(0) = 4$	-	-	-	4	1
4	$0 \cdot 4 + f_1^*(4) = 0$	$1 \cdot 4 + f_1^*(1) = 4$	-	-	-	4	1
5	$0 \cdot 4 + f_1^*(5) = 9$	$1 \cdot 4 + f_1^*(2) = 4$	-	-	-	9	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

That is, if

> $f_2^*(\widehat{W})$ denote the maximum value if we only have products 1 and 2, and the knapsack capacity is $\widehat{W}$,

then

$\widehat{W}$	$f_2^*(\widehat{W})$	x_2^*
0	$0 \cdot 4 + f_1^*(0) = 0$	0
1	$0 \cdot 4 + f_1^*(1) = 0$	0
2	$0 \cdot 4 + f_1^*(2) = 0$	0
3	$\max\{0 \cdot 4 + f_1^*(3), 1 \cdot 4 + f_1^*(0)\} = 4$	1
4	$\max\{0 \cdot 4 + f_1^*(4), 1 \cdot 4 + f_1^*(1)\} = 4$	1
5	$\max\{0 \cdot 4 + f_1^*(5), 1 \cdot 4 + f_1^*(2)\} = 9$	0
6	$\max\{0 \cdot 4 + f_1^*(6), 1 \cdot 4 + f_1^*(3), 2 \cdot 4 + f_1^*(0)\} = 9$	0
7	$\max\{0 \cdot 4 + f_1^*(7), 1 \cdot 4 + f_1^*(4), 2 \cdot 4 + f_1^*(1)\} = 9$	0
8	$\max\{0 \cdot 4 + f_1^*(8), 1 \cdot 4 + f_1^*(5), 2 \cdot 4 + f_1^*(2)\} = 13$	1
9	$\max\{0 \cdot 4 + f_1^*(9), 1 \cdot 4 + f_1^*(6), 2 \cdot 4 + f_1^*(3), 3 \cdot 4 + f_1^*(0)\} = 13$	1
10	$\max\{0 \cdot 4 + f_1^*(10), 1 \cdot 4 + f_1^*(7), 2 \cdot 4 + f_1^*(4), 3 \cdot 4 + f_1^*(1)\} = 18$	0
11	$\max\{0 \cdot 4 + f_1^*(11), 1 \cdot 4 + f_1^*(8), 2 \cdot 4 + f_1^*(5), 3 \cdot 4 + f_1^*(2)\} = 18$	0
12	$\max\{0 \cdot 4 + f_1^*(12), 1 \cdot 4 + f_1^*(9), 2 \cdot 4 + f_1^*(6), 3 \cdot 4 + f_1^*(3), 4 \cdot 4 + f_1^*(0)\} = 18$	0
13	$\max\{0 \cdot 4 + f_1^*(13), 1 \cdot 4 + f_1^*(10), 2 \cdot 4 + f_1^*(7), 3 \cdot 4 + f_1^*(4), 4 \cdot 4 + f_1^*(1)\} = 22$	1

More generally, let

> $f_k^*(\widehat{W})$ denote the maximum value if we only have products 1, 2, $\ldots, k$, and the knapsack capacity is $\widehat{W}$.

3. Next, suppose that we have three products: products 1, 2, and 3. As we did in the previous step, we will consider the possible number of copies of product 3 that we can fit into a knapsack of remaining weight capacity of $\widehat{W}$. In the table below, we consider this for various possible values of $\widehat{W}$, from 0 to 13.

$\widehat{W}$	Max value, if $x_3 = 0$	Max value if $x_3 = 1$	Max value if $x_3 = 2$	Max value if $x_2 = 3$	Max value if $x_2 = 4$	$f_3^*(\widehat{W})$	x_3^*
0	$0 \cdot 4 + f_1^*(0) = 0$	-	-	-	-	0	0
1	$0 \cdot 4 + f_1^*(1) = 0$	-	-	-	-	0	0
2	$0 \cdot 4 + f_1^*(2) = 0$	-	-	-	-	0	0
3	$0 \cdot 4 + f_1^*(3) = 0$	$1 \cdot 4 + f_1^*(0) = 4$	-	-	-	4	1
4	$0 \cdot 4 + f_1^*(4) = 0$	$1 \cdot 4 + f_1^*(1) = 4$	-	-	-	4	1
5	$0 \cdot 4 + f_1^*(5) = 9$	$1 \cdot 4 + f_1^*(2) = 4$	-	-	-	9	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Using the method above, complete the following table.

$\widehat{W}$	$f_3^*(\widehat{W})$	x_3^*
0	$0 \cdot 3 + f_2^*(0) = 0$	0
1	$0 \cdot 3 + f_2^*(1) = 0$	0
2	$\max\{0 \cdot 3 + f_2^*(2), 1 \cdot 3 + f_2^*(0)\} =$	
3	$\max\{0 \cdot 3 + f_2^*(3), 1 \cdot 3 + f_2^*(1)\} =$	
4	$\max\{0 \cdot 3 + f_2^*(4), 1 \cdot 3 + f_2^*(2), 2 \cdot 3 + f_2^*(0)\} =$	
5	$\max\{0 \cdot 3 + f_2^*(5), 1 \cdot 3 + f_2^*(3), 2 \cdot 3 + f_2^*(1)\} =$	
6	$\max\{ \qquad \} =$	
7		
8		
9		
10		
11		
12		
13		

Formalizing the DP approach above:

- What are the stages in this DP formulation of the knapsack problem?

- What are the states in this DP formulation of the knapsack problem?

- Given the above stages and states information, describe in words the decisions that we consider at each state in each stage.

- What is the word-description of $f_k^*(\widehat{W})$?

- Which values of k and $\widehat{W}$ did we compute $f_k^*(\widehat{W})$ first? These provide our "boundary conditions".

- Give a recurrence relation for $f_k^*(\widehat{W})$ based on our method above.

- For what value of $\widehat{W}$ does $f_k^*(\widehat{W})$ corresponds to the maximum value of our original knapsack problem?

3 The Distribution of Effort Problem

A government space agency is conducting research project on an engineering problem that must be solved before people can safely fly to Mars. Three independent research teams are currently trying three different approaches to solve this problem.

The probability that team 1 will fail to solve this problem is 0.40, the probability that team 2 will fail is 0.60, and the probability that team 3 will fail is 0.80. We say that the project fails if all three teams fail. So, currently, the probability that the project fails is $0.40 \cdot 0.60 \cdot 0.80 = 0.192$.

The space agency decides to assign a total of two new scientist to the project. The two scientists can be assigned to together or separately to any of the teams. The following table gives the new probabilities of failure if 0, 1, or 2 new scientist is assigned to each team. Our task is to use a dynamic programming approach to decide how many new scientist to assign to each team such that the probability of project failure is minimized.

Number of new scientists	Probability of failure		
	Team 1	Team 2	Team 3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30