Recitation 14

In this recitation, we will implement the interior-point method for solving a linear programming problem

Name and NetID:
Section:

Consider the following linear programming problem:

$$\begin{aligned} \max \quad & 2x_1 + x_2 & \\ s.t. \quad & 5x_1 + x_2 & \le 10 \\ & x_1 & \ge 0 \\ & x_2 & \ge 0 \end{aligned}$$

1. Write down the barrier function, $f_\mu(x_1, x_2)$ for the above linear programming problem.

2. We would like to find an optimal solution to the unconstrained optimization problem:

   $$\max f_\mu(x_1, x_2).$$

   Suppose that we let $x(\mu)$ denote the optimal solution to this optimization problem, write down the necessary optimality condition that $x(\mu)$ must satisfy.

In the next few steps, we will solve for $x(\mu)$ using Newton's Method.

(a) We can rewrite the optimality conditions above in the form of:

$$\begin{aligned} g_1(x_1, x_2) &= 0 \quad (1) \\ g_2(x_1, x_2) &= 0 \quad (2) \end{aligned}$$

for some functions $g_1(x_1, x_2), g_2(x_1, x_2)$ that are polynomials in x_1 and x_2. Write down what $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are.

(b) Suppose that we would like to obtain linear approximations for g_1 and g_2 at the point $(\bar{x}_1, \bar{x}_2)$. These linear approximations are:

$$\begin{aligned} l_1(x_1, x_2) &= g_1(\bar{x}_1, \bar{x}_2) + \nabla g_1(\bar{x}_1, \bar{x}_2)^T (x - \bar{x}) \\ l_2(x_1, x_2) &= g_2(\bar{x}_1, \bar{x}_2) + \nabla g_2(\bar{x}_1, \bar{x}_2)^T (x - \bar{x}). \end{aligned}$$

Write down what $\nabla g_1(\bar{x}_1, \bar{x}_2)$ and $\nabla g_1(\bar{x}_1, \bar{x}_2)$ are, in terms of $\bar{x}_1, \bar{x}_2$, and μ.

(c) Suppose that our initial value for μ is $\mu = 5$. Use one Newton's method iteration to (approximately) find $x(\mu) = x(5)$, with a starting point $(\bar{x}_1, \bar{x}_2) = (1, 1)$.

(d) More generally, the system of linear equations

$$\begin{aligned} l_1(x_1, x_2) &= 0 \\ l_2(x_1, x_2) &= 0 \end{aligned}$$

can be written as

$$\begin{pmatrix} g_1(\bar{x}_1, \bar{x}_2) \\ g_2(\bar{x}_1, \bar{x}_2) \end{pmatrix} + \begin{pmatrix} \nabla g_1(\bar{x}_1, \bar{x}_2)^T \\ \nabla g_2(\bar{x}_1, \bar{x}_2)^T \end{pmatrix} (x - \bar{x}) = 0.$$

So, if our current Newton's Method iterate is $\bar{x} = (\bar{x}_1, \bar{x}_2)$, then solving for x, the next iterate:

$$x = \bar{x} - M^{-1} v$$

where M is the matrix:

$$M = \begin{pmatrix} \nabla g_1(\bar{x}_1, \bar{x}_2)^T \\ \nabla g_2(\bar{x}_1, \bar{x}_2)^T \end{pmatrix}$$

and v is the vector:

$$v = \begin{pmatrix} g_1(\bar{x}_1, \bar{x}_2) \\ g_2(\bar{x}_1, \bar{x}_2) \end{pmatrix}.$$

Write down the entries of M and v in terms of $\bar{x}_1, \bar{x}_2, \mu$:

$$M = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix},$$

$$v = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

3. Next, we will reduce μ from 5 to a smaller value, call it μ_2, and again use one iteration of Newton's method to (approximately) solve for $x(\mu_2)$.

   Suppose that (for this exercise) we choose $\mu_2 = 3$. As a starting point for our Newton's Method iteration, use the approximation to $x(5)$ which we found in 1(c).

4. One more time: reduce μ yet again. Let $\mu_3 = 1.8$, and use one iteration of Newton's method to (approximately) solve for $x(\mu_3)$. As a starting point for our Newton's method, use the approximation to $x(3)$ which we found above.

5. Sketch the feasible region of our LP, and indicate in your sketch (the approximations of) the points $x(5), x(3), x(1.8)$ which you found in parts 1(c), 2, and 3 above. Also indicate where the actual LP solution is.

6. Summarize our new algorithm for solving the above linear program.

   Step 1: Start with an initial $\mu^{(0)} = \ldots$, $x^{(0)} = (x_1, x_2) = (\ldots, \ldots)$.

   Step 2: At the ith iteration: ...

7. Download the zip file under "Recitation 14" from blackboard. This zip file contains a few Matlab files that uses our new algorithm to solve our LP example. Unzip the contents into a folder on your computer, and examime the script "Rec14.m".

   If you run this file, it will produce a plot that traces the points $x(\mu_i)$ where $\mu_1 = 5$, and $\mu_{i+1} = \mu_i/\beta$ with $\beta = 5/3$.

   Try to understand the gist of the script, then play around by changing any of the following values:

   - `NMiterations`
   - `IPMiterations`
   - `beta`
   - the initial value of `mu`
   - the initial `x1, x2`

   How does the behaviour of the iterates change as you change these values?