Recitation 3

In this recitation, we will consider the maximum flow problem again in connection with the minimum-cut problem. We will use AMPL and recall some results from linear programming duality.

You can work in pairs or individually. Answer the questions in the blank space provided, and turn this handout in to your section TA. If you work with a partner, you and your partner need only turn in one copy.

Name and NetID:
Section:

1 The Maximum-flow and the minimum-cut problems

1. Open the AMPL model file `maxflow.mod` from the `MODELS` folder under `amplcml`. Change this model so that the capacity constraints are explicit constraints of the model, rather than just upper bounds on the variable `Flow`. (You can save the modified file as a new `.mod` file, if you like.)

2. Consider the corresponding data file, `maxflow.dat` (reproduced here). In the space provided below, draw the graph for this input.

```
set nodes := s a b c d e t ;

param orig := s ;
param dest := t ;

param:  arcs:  cap :=
         s a    1     s b    4     s c    6
         a b    3     a d    4
         b a    2     b d    3     b e    1
         c e    4     d t    9     e t    4 ;
```

3. Write out the linear programming formulation of the maximum flow problem for this input. There are 11 variables, 5 equality constraints, and 11 inequality constraints. Write the flow conservation constraints in the form "flow out - flow in equals 0". For each variable, try to be very careful to keep all occurrences of this variable in the various constraints lined up in the same column, as if you were just writing the coefficient matrix A in a system of equations $Ax = b$.

4. Write out the dual to this linear program. For each node $k \neq s, t$, let v_k be the dual variable corresponding to the flow conservation constraint at K, and for each arc (i, j), let w_{ij} denote the dual variable corresponding to the capacity constraint for arc (i, j).

   [Recall that each equality constraint has a dual variable that is unrestricted in sign, whereas each inequality constraint has a dual variable that is non-negative. Furthermore, for each non-negative variable, the corresponding constraint is an inequality constraint.]

5. Recall *the complementary slackness condition* that hold for pairs of optimal solutions to the primal and dual linear programs. If one of the variables w_{ij} is positive for the optimal dual solution, what must be true for the optimal primal solution?

6. Now use AMPL with the given `maxflow.mod` and `maxflow.dat` files to solve this input to the maximum flow problem. Recall that you can also display the optimal solution to the dual linear program by displaying the (primal) constraints, instead of displaying the (primal) variables.

   What is the optimal solution to the dual linear program? Use this dual solution to compute the optimal dual objective function value by hand. (Yes, this must equal the maximum flow value, but compute it out explictly by hand anyway.)

7. How do you suspect that the optimal dual solution corresponds to the minimum cut?

2 Using Maxflow to determine the feasibility of Mincost Flow problems

Suppose that we are given a particular input to the minimum-cost flow problem. How do we know whether it has a feasible solution? It turns out that we can use the maximum-flow model to determine whether there exists a feasible solution!

- First, we will create an input to the maximum-flow problem based on the minimum-cost flow input that is given to us, as follows:
  - Add a source node s and a sink node t
  - for each supply node i (i.e. node i with supply value $b_i > 0$), introduce an edge (s, i) with capacity b_i,
  - for each demand node j (i.e. node i with supply value $b_i < 0$), introduce an edge (j, t) with capacity $-b_i$.
- Then, there is a feasible solution for the minimum-cost flow problem if and only if the maximum flow value is equal to the total of the positive supply values: $\sum_{i \in N \ s.t. \ b_i > 0} b_i$.

Here is a data file `Rec3.dat` for an input to the minimum-cost flow problem, which you will find available on blackboard, under the "Recitations" folder. (Observe that the total demand is equal to the total supply.)

```
set nodes := a b c d e f g;
set edges := (a, d) (b, d) (c, d) (c, e) (d, e) (d, f) (e, f) (e, g);

param:  capacity cost :=
a d  20  4
b d  10  3
c d  10  0
c e  10  10
d e   5  2
d f  20  5
e f  10  7
e g  20  9;

param supply:=
a 15
b 10
c 9
d 0
e 0
f -19
g -15;
```

1. Modify the data file that we provide to create an input to the maximum flow problem model that you used in the first part of this recitation exercise. That is, this data file should be compatible for use with `maxflow.mod` that we saw before.

2. Now solve the maximum flow problem for this input, using the `maxflow.mod` model file and the data file you have just modified above. What does the value of the maximum flow say about whether there is a feasible solution to the transshipment input problem?

3. Now use the optimal *dual* solution to determine the minimum cut in this graph. What is the minimum cut for this maximum flow problem?

4. Study how this cut relates to the original input to the minimum-cost network flow problem. If you understand this completely, you will be able to realize that you have just proved an important theorem about feasible solutions to this problem.

   Theorem 1. *For any input to the minimum-cost network flow problem in which the total supply is equal to the total demand, there exists a feasible solution if and only if, for any partition of the nodes of this input into sets S and T, the total capacity of the arcs going from a node in S to a node in T...*

   Complete the theorem above. Why could you conclude that it is true?