Recitation 10

In this recitation, we will solve a small integer program using the Gomory Cutting Planes method. **You do not need to hand in this sheet. Only hand in the shorter recitation handout to your TA.**

1 The problem

Example. We will solve the following integer programming problem, which we will refer to as (IP).

$$\begin{aligned} \max \quad & 5x_1 + 7x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 4 \\ & 5x_1 + 6x_2 \leq 15 \\ & x_1, x_2 \geq 0, \text{ integers.} \end{aligned} \qquad (IP)$$

One big idea that is the basis of many methods for solving integer programs is: using the linear programming relaxation of an integer program to help us solve the integer program itself.

So, let us consider the LP-relaxation of (IP):

$$\begin{aligned} \max \quad & 5x_1 + 7x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 4 \\ & 5x_1 + 6x_2 \leq 15 \\ & x_1, x_2 \geq 0. \end{aligned} \qquad (LP_1)$$

Let us refer to the above linear program as (LP_1).

Question 1. Rewrite the (LP_1) with added slack variables, so that we only have equality constraints.

Question 2. Using AMPL, solve (LP_1) with the added slack variables.

[Use the sample `.mod` file that is given on blackboard (under Recitation 10) and make the necessary changes to this model file. You do not need to write a data file.]

1. What is the optimal solution that you obtain?

2. For the optimal solution that you obtained above, which are the **basic variables**, and which are the **nonbasic variables**?

3. Hence, what is the corresponding **basis** for this optimal solution?

2 Outline of the Gomory Cutting Plane method

The following is the big picture of the Gomory Cutting Planes method for solving (IP):

The Gomory Cutting Planes Method

0. Solve $(LP1)$. Suppose that $x^*_{LP_1}$ is an optimal solution to (LP_1).
1. At the kth iteration: Solve (LP_k). Suppose that $x^*_{LP_k}$ is an optimal solution to (LP_k).
2. If $x^*_{LPk} = (x_1, x_2)$, and both x_1 and x_2 are integer-valued, then $x^*_{LP_k}$ is an optimal solution to (IP). (We're done!)

   Otherwise, if $x^*_{LP_k} = (x_1, x_2)$ and either x_1 or x_2 is not integer-valued, then we add an additional constraint (this constraint is the Gomory cutting plane) to (LP_k). This gives us a new linear program which we will denote (LP_{k+1}). Go to step 1 and carry out the $(k+1)$th iteration.

Remark. Note that the outline above does not specify what constraint is to be added in step 2. This is because we have to use the simplex tableau to formulate the additional constraint. We will first review the simplex method in the following section.

3 A brief review of the simplex method

Let us solve $(LP1)$ using the simplex method. The following is $(LP1)$ with slack variables x_3, x_4 added:

$$\begin{array}{rc}
\max & 5x_1 + 7x_2 \\
s.t. & x_1 + 2x_2 + x_3 = 4 \\
& 5x_1 + 6x_2 + x_4 = 15 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{array} \qquad (LP_1)$$

First, recall the algorithm:

The Simplex Method

0. Choose an initial feasible basis and setup a simplex tableau with respect to this basis.

   For our problem, $\{x_3, x_4\}$ is a feasible basis that we can choose as our initial feasible basis.

1. At the kth iteration: given the current simplex tableau:

   (a) If all reduced costs are zero or negative, then we have obtained an optimal solution.

   (b) Otherwise, choose a variable x_i with positive reduced cost. The variable x_i will enter the basis.

   Then, carry out the minimum-ratio test to choose a variable that will leave the basis.

   At the end of this step, we have a new basis. So, we update the simplex tableau with respect to this basis. Repeat Step 1.

Then, let us carry out the simplex method for our example.

Initial tableau. We choose the basis to be $\{x_3, x_4\}$. The initial simplex tableau corresponding to this basis:

	x_1	x_2	x_3	x_4	
Reduced costs $= \bar{c}_j$	5	7	0	0	rhs $= \bar{b}_i$
x_3	1	2	1	0	4
x_4	5	6	0	1	15

This simplex tableau corresponds to the solution $(x_1, x_2, x_3, x_4) = (0, 0, 4, 15)$, which is a basic feasible solution for $(LP1)$.

The basic variables are x_3, x_4: these are the variables that are in the basis.

The nonbasic variables are x_1, x_2: these are variables that are not in the basis. Their values are zero.

Question 3. If we have a linear program with m equality constraints and n nonnegative variables, how many variables should be in the basis? How many variables are not in the basis?

Observe the columns of the basic variables in the simplex tableau above. What is true about the entries of each of these columns?

Iteration 1 In the current tableau above, the reduced costs are: $\bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = (5, 7, 0, 0)$.

Hence, either x_1 or x_2 can enter the basis, because $\bar{c}_1 = 5 > 0$ and $\bar{c}_2 = 7 > 0$. Suppose we let x_2 enter the basis. Then, we need to determine which one of x_3 or x_4 should leave the basis, using the minimum-ratio rule: For each row, take the ratio of the right hand side to the entry of column of x_2 in that row:

$$\min \left\{ \underbrace{\frac{4}{2}}_{\text{row 1}}, \underbrace{\frac{15}{6}}_{\text{row 2}} \right\} = \frac{4}{2} = 2.$$

Hence, the variable that leaves the basis is the row 1 basic variable, namely x_3.

So, adding the variable x_2 and removing the variable x_x from the old basis $\{x_3, x_4\}$, we see that the new basis is: $\{x_2, x_4\}$.

We claim that the new tableau with respect to the basis $\{x_2, x_4\}$ is:

	x_1	x_2	x_3	x_4	
Reduced costs $= \bar{c}_j$	1.5	0	-3.5	0	rhs $= \bar{b}_i$
x_2	0.5	1	0.5	0	2
x_4	2	0	-3	1	3

Remark. How did we obtain this tableau?

- Our constraint matrix is:
$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{bmatrix},$$
and the current basis is $\{x_2, x_4\}$. So, the basis matrix is the matrix B with the 2nd and 4th columns of A.
$$B = \begin{bmatrix} 2 & 0 \\ 6 & 1 \end{bmatrix}.$$
- Note that the inverse of B is
$$B^{-1} = \begin{bmatrix} 0.5 & 0 \\ -3 & 1 \end{bmatrix}.$$
- Then, the entries of the tableau can be obtained from the matrix:
$$\bar{A} = B^{-1}A = \begin{bmatrix} 0.5 & 1 & 0.5 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}.$$
- The right hand side of the tableau is obtained from:
$$\bar{b} = B^{-1}b = B^{-1}\begin{bmatrix} 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
- The top row of the tableau (i.e., the **reduced cost** with respect to the current basis) is obtained from:
$$\begin{aligned} \bar{c} &= c - c_B B^{-1} A \\ &= (5, 7, 0, 0) - (7, 0) B^{-1} A \\ &= (1.5, 0, -3.5, 0) \end{aligned}$$

- Given any basis $\{x_i, x_j\}$ (in this example, any feasible basis has $m = 2$ variables because we have $m = 2$ constraints), we can obtain the corresponding tableau using the method above.

Iteration 2. In the current tableau above, the reduced costs are: $\bar{c} = (1.5, 0, -3.5, 0)$. Hence, x_1 can enter the basis, because $\bar{c}_1 = 1.5 > 0$.

Question 4. Using the minimum-ratio test, which variable leaves the basis?

What is the new basis?

The following is the new tableau with respect to the new basis (**complete the following tableau**).

	x_1	x_2	x_3	x_4	
Reduced costs $= \bar{c}_j$	0	0			rhs $= \bar{b}_i$
x_2	0	1			
x_1	1	0			

Iteration 3. With respect to the current tableau above:

Question 5. What can you say about the above tableau? If it does not correspond to an optimal solution, specify which variable should enter the basis and which should leave. If it does correspond to an optimal solution, explain why.

4 Using the Gomory Cutting Planes method

In this section, we will carry out the following steps:

The Gomory Cutting Planes Method

0. Solve $(LP1)$. Suppose that $x^*_{LP_1}$ is an optimal solution to (LP_1).

1. At the kth iteration: Solve (LP_k). Suppose that $x^*_{LP_k}$ is an optimal solution to (LP_k).

2. If $x^*_{LPk} = (x_1, x_2)$, and both x_1 and x_2 are integer-valued, then $x^*_{LP_k}$ is an optimal solution to (IP). (We're done!)

   Otherwise, if $x^*_{LP_k} = (x_1, x_2)$ and either x_1 or x_2 is not integer-valued, then we add an additional constraint (this constraint is the Gomory cutting plane) to (LP_k). This gives us a new linear program which we will denote (LP_{k+1}). Go to step 1 and carry out the $(k+1)$th iteration.

   The constraint that we add is as follows: suppose the right hand side of row i of the simplex tableau is not integer-valued, then we add the constraint

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor , \tag{1}$$

   where

$$\begin{aligned} N &= \text{denote the set of the indices of the nonbasic variables} \\ \bar{b}_i &= \text{denote the right hand side of row } i \text{ of the simplex tableau} \\ \bar{a}_{ij} &= \text{denote the entry in row } i \text{ of the column of variable } x_j \text{ in the simplex tableau} \end{aligned}$$

We start with step 0:

Initialize. We solve $(LP1)$ using the simplex method, which we did in the previous section.

We obtain an optimal solution $x = (1.25, 1.5, 0, 0)$ for $(LP1)$, which corresponds to the tableau:

	x_1	x_2	x_3	x_4	
Reduced costs $= \bar{c}_j$	0	0	-1.25	-0.75	rhs $= \bar{b}_i$
x_2	0	1	1.25	-0.25	1.25
x_1	1	0	-1.5	0.5	1.5

Note that in this tableau, the basis is $\{x_2, x_1\}$. So, the set of indices of the nonbasic variables is $N = \{3, 4\}$.

Iteration 1. This solution is not a feasible solution for IP, since both x_1, x_2 are not integer-valued. So, suppose that we choose row $i = 2$, which in the above tableau corresponds to the row of x_1.

Question 6. Then, following (1), the Gomory cutting plane is:

Adding a slack variable x_5, the Gomory cutting plane is:

Hence, the second linear programming that we solve is (call it (LP_2)):

$$\begin{array}{rl} \max & 5x_1 + 7x_2 \\ s.t. & x_1 + 2x_2 + x_3 = 4 \\ & 5x_1 + 6x_2 + x_4 = 15 \\ & \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array} \qquad (LP_2)$$

What we have to do next is to find an optimal solution for (LP_2).

Instead of solving (LP_2) from scratch, we can use the optimal simplex tableau of (LP_1) as a starting point.

Adding the new constraint to the optimal simplex tableau of (LP_1), we obtain (**complete the following tableau**):

	x_1	x_2	x_3	x_4	x_5	
Reduced costs $= \bar{c}_j$	0	0	-1.25	-0.75		rhs $= \bar{b}_i$
x_2	0	1	1.25	-0.25		1.25
x_1	1	0	-1.5	0.5		1.5
x_5					1	

Note that the column of x_1 is no longer like a column of the identity matrix. **Update the simplex tableau** so that the last entry of the column of x_1 is zero (complete the following tableau):

	x_1	x_2	x_3	x_4	x_5	
Reduced costs $= \bar{c}_j$	0	0	-1.25	-0.75	??	rhs $= \bar{b}_i$
x_2	0	1	1.25	-0.25	0	1.25
x_1	1	0	-1.5	0.5	0	1.5
x_5	0	0	?	-0.5	1	-0.5

Since the right hand side of the third row is negative, this means that the current basis does not correspond to a feasible basic solution. Hence, we need to carry out the **dual simplex**

method: After identifying a row with negative left hand side (i.e., the third row), we carry out the minimum-ratio test for the dual simplex method:

$$\min\left\{\underbrace{\frac{-1.25}{?}}_{\text{col 3}},\underbrace{\frac{-0.75}{-0.5}}_{\text{col 4}}\right\}=\underbrace{\frac{-0.75}{-0.5}}_{\text{col 4}}=1.5,$$

where the minimum is achieved by column 4 (colum 4 corresponds to x_4).

So, we pivot: the old basis was $\{x_2, x_1, x_5\}$, x_5 leaves the basis and x_4 enters the basis. So, the new basis is: $\{x_2, x_1, x_4\}$. The corresponding tableau is:

	x_1	x_2	x_3	x_4	x_5	
Reduced costs $=\bar{c}_j$	0	0	-0.5	0	-1.5	rhs $=\bar{b}_i$
x_2	0	1	1.5	0	-0.5	1.5
x_1	1	0	-2	0	1	1
x_4	0	0	1	1	-2	1

Question 7. This tableau corresponds to an optimal solution for (LP_2). What is this solution?

Explain why this solution is optimal.

Note that in this tableau, the basis is $\{x_2, x_1, x_4\}$. So, the set of indices of the nonbasic variables is $N = \{3, 5\}$.

Iteration 2. The tableau above does not correspond to a feasible solution for IP, since $x_2 = 1.5$ is not integer-valued. Because x_2 is on the first row of the tableau above, choose row $i = 1$ to get another Gomory cutting plane.

Question 8. Then, following (1), the Gomory cutting plane is:

Adding a slack variable x_6, the Gomory cutting plane is:

Hence, the second linear programming that we solve is (call it (LP_3)):

$$\begin{array}{lll} \max & 5x_1 + 7x_2 & \\ s.t. & x_1 + 2x_2 + x_3 = 4 & (LP_3) \\ & 5x_1 + 6x_2 + x_4 = 15 & \\ & & \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. & \end{array}$$

Instead of solving (LP_3) from scratch, we can use the optimal simplex tableau of (LP_2) as a starting point.

Adding the new constraint to the optimal simplex tableau of (LP_2), we obtain (**complete the following tableau**):

	x_1	x_2	x_3	x_4	x_5	x_6	
Reduced costs $= \bar{c}_j$	0	0	-0.5	0	-1.5	0	rhs $= \bar{b}_i$
x_2	0	1	1.5	0	-0.5	0	1.5
x_1	1	0	-2	0	1	0	1
x_4	0	0	1	1	-2	0	1
x_6	0	1	1	0	-1	1	1

Note that the column of x_2 is no longer like a column of the identity matrix. **Update the simplex tableau** so that the last entry of the column of x_2 is zero (complete the following tableau):

	x_1	x_2	x_3	x_4	x_5	x_6	
Reduced costs $= \bar{c}_j$	0	0	-0.5	0	-1.5	0	rhs $= \bar{b}_i$
x_2	0	1	1.5	0	-0.5	0	1.5
x_1	1	0	-2	0	1	0	1
x_4	0	0	1	1	-2	0	1
x_6	0	0	-0.5	0	-0.5	1	-0.5

Since the right hand side of the third row is negative, this means that the current basis does not correspond to a feasible basic solution. Hence, we need to carry out the **dual simplex**

method: After identifying a row with negative left hand side (i.e., the fourth row), we carry out the minimum-ratio test for the dual simplex method:

$$\min \left\{ \underbrace{\frac{-0.5}{-0.5}}_{\text{col 3}}, \underbrace{\frac{-1.5}{-0.5}}_{\text{col 5}} \right\} = \underbrace{\frac{-0.5}{-0.5}}_{\text{col 3}} = 1,$$

where the minimum is achieved by column 3 (colum 3 corresponds to x_3).

So, we pivot: the old basis was $\{x_2, x_1, x_4, x_6\}$, x_6 leaves the basis and x_3 enters the basis. So, the new basis is: $\{x_2, x_1, x_4, x_3\}$. The corresponding table is:

	x_1	x_2	x_3	x_4	x_5	x_6	
Reduced costs $= \bar{c}_j$	0	0	0	0	-1	-1	rhs $= \bar{b}_i$
x_2	0	1	0	0	-2	3	0
x_1	1	0	0	0	3	-4	3
x_4	0	0	0	1	-3	2	0
x_3	0	0	1	0	1	-2	1

Question 9. This tableau corresponds to an optimal solution for (LP_3). What is this solution?

Explain why this solution is optimal.

Iteration 3. The tableau above does correspond to a feasible solution for (IP).

Question 10. Why?

What is the optimal solution to the integer program (IP)?