Lecture 18: March 28, 2013

In this lecture, we will pick up where we left off last time with integer programming formulation "tricks" using binary decision variables.

1 Useful integer programming formulation tricks

In the nonlinear knapsack example above, the objective function is not a linear function of the number of units of the products that are taken. However, by using binary indicator decision variables, we were able to formulate the problem as an integer program.

1.1 Some simple integer programming constraints

1. Suppose that n activities are available, and we were to select exactly one of these activities. Let the binary decision variable x_i be an indicator of whether we select activity i. That is, if $x_i = 1$, we select activity i, but if $x_i = 0$, we don't. Hence, the constraint that force us to select exactly one out of the n available activities is:
$$x_1 + x_2 + \ldots + x_n = 1.$$
2. If instead, we want to select at least two of the activities, then we use the constraint:
$$x_1 + x_2 + \ldots + x_n \geq 2.$$
3. If we want a constraint to express that "activity 3 is possible only if both activities 1 and 2 are selected", then we add the following two constraints:
$$x_3 \leq x_1,$$
$$x_3 \leq x_2.$$
4. Suppose that you can only select activity 3 if at least one of activities 1 or 2 are selected:
$$x_3 \leq x_1 + x_2$$

(This is where we stopped last lecture.)

The above formulation using binary decision variables can be used more generally. To illustrate this, we first consider the following basic integer program:

Example 1.

$$\begin{aligned}
\max \quad & 3x_A + 2x_B + 1x_C + 2x_D + 3x_E \\
s.t. \quad & 4x_A + 2x_B + 5x_C + 3x_D + 2x_E \leq 35 && (1)\\
& 2x_A + 4x_B + 3x_C + 4x_D + 1x_E \leq 40 && (2)\\
& 1x_A + 1x_B + 3x_C + 2x_D + 2x_E = 30 && (3)\\
& 4x_A - 3x_B + 5x_C - 2x_D + 3x_E \geq 50 && (4)\\
& 0x_A + 0x_B - 4x_C + 2x_D + 2x_E \geq 10 && (5)\\
& x_A, x_B, x_C, x_D, x_E \geq 0 \\
& x_A, x_B, x_C, x_D, x_E \text{ integers.}
\end{aligned}$$

In the following sections, we will introduce a few modifications to this problem, which we can incorporate into our integer program after introducing new binary decision variables.

1.2 "At least k constraints out of m must hold" and the "big-M" method

Consider the problem in the example above. Suppose that out of the five constraints, it turns out that only at least 3 of them have to be satisfied.
Formulation:

- Consider new binary decision variables z_i for $i = 1, \ldots, 5$. We interpret these decision variables as follows: if $z_i = 1$ then the constraint (i) is satisfied, and if $z_i = 0$, then the constraint (i) is not necessarily satisfied.
- Add the constraint:
$$z_1 + z_2 + z_3 + z_4 + z_5 \geq 3,$$
requiring that at least 3 out of the five constraints must be satisfied.
- Modify the five constraints as follows:
$$\begin{aligned}
4x_A + 2x_B + 5x_C + 3x_D + 2x_E &\leq 35 + M(1 - z_1) \\
2x_A + 4x_B + 3x_C + 4x_D + 1x_E &\leq 40 + M(1 - z_2) \\
1x_A + 1x_B + 3x_C + 2x_D + 2x_E &\leq 30 + M(1 - z_3) \\
1x_A + 1x_B + 3x_C + 2x_D + 2x_E &\geq 30 - M(1 - z_3) \\
4x_A + 3x_B + 5x_C - 2x_D + 3x_E &\geq 50 - M(1 - z_4) \\
0x_A + 0x_B - 4x_C + 2x_D + 2x_E &\geq 10 - M(1 - z_5),
\end{aligned}$$
where, M is a sufficiently large positive number. (We can think of M as "positive infinity" but in our example, $M = 1000$ is sufficiently large.)

Hence, the modified integer program is:

$$\begin{array}{rc}
\max & 3x_A + 2x_B + 1x_C + 2x_D + 3x_E \\
s.t. & 4x_A + 2x_B + 5x_C + 3x_D + 2x_E \leq 35 + M(1 - z_1) \\
& 2x_A + 4x_B + 3x_C + 4x_D + 1x_E \geq 40 + M(1 - z_2) \\
& 1x_A + 1x_B + 3x_C + 2x_D + 2x_E \leq 30 + M(1 - z_3) \\
& 1x_A + 1x_B + 3x_C + 2x_D + 2x_E \geq 30 - M(1 - z_3) \\
& 4x_A + 3x_B + 5x_C - 2x_D + 3x_E \geq 50 - M(1 - z_4) \\
& 0x_A + 0x_B - 4x_C + 2x_D + 2x_E \geq 10 - M(1 - z_5) \\
& z_1 + z_2 + z_3 + z_4 + z_5 \geq 3 \\
& x_A, x_B, x_C, x_D, x_E \geq 0 \\
& x_A, x_B, x_C, x_D, x_E \text{ integers,} \\
& 0 \leq z_i \leq 1, \text{ integer, } \quad \forall i \in \{1, \ldots, 5\}
\end{array}$$

Note that using similar methods, we can handle conditions that require "exactly k out of m constraints must hold" or "at most k out of m constraints must hold.

Question. If the original five constraints had been "greater than" constraints instead of "less than" constraints, how would we modify the integer program to incorporate the condition that at least 3 out of the 5 "greater than" constraints must hold?

1.3 "If ... then ..." constraints and the "big-M" method

Consider again the original problem in Example 4. Suppose that we would like to incorporate the condition that if constraints (1) and (2) are satisfied, then constraint (3) have to be sastisfied. (But if not both of (1) and (2) are satisfied, then (3) does not have to be satisfied)
Formulation:

- Consider new binary decision variables z_1, z_2, z_3. We interpret $z_i = 1$ to indicate that constraint (i) holds, and $w_i = 0$ to indicate that constraint (i) does not necessarily hold.
- Next, we add the constraint express the condition that if constraints (1) and (2) are both satisfied, then (3) does not have to be satisfied:
$$z_3 \leq z_1,$$
$$z_3 \leq z_2.$$
- Then, we modify constraints (1), (2), and (3) as follows:
$$4x_A + 2x_B + 5x_C + 3x_D + 2x_E \leq 35 + M(1 - z_1)$$
$$2x_A + 4x_B + 3x_C + 4x_D + 1x_E \leq 40 + M(1 - z_2)$$
$$1x_A + 1x_B + 3x_C + 2x_D + 2x_E \leq 30 + M(1 - z_3)$$
$$1x_A + 1x_B + 3x_C + 2x_D + 2x_E \geq 30 - M(1 - z_3)$$
where M is, again, a sufficiently large positive number.

Hence, the modified integer program is:

$$\begin{array}{rl}
\max & 3x_A + 2x_B + 1x_C + 2x_D + 3x_E \\
s.t. & 4x_A + 2x_B + 5x_C + 3x_D + 2x_E \leq 35 + M(1 - z_1) \\
& 2x_A + 4x_B + 3x_C + 4x_D + 1x_E \leq 40 + M(1 - z_2) \\
& 1x_A + 1x_B + 3x_C + 2x_D + 2x_E \leq 30 + M(1 - z_3) \\
& 1x_A + 1x_B + 3x_C + 2x_D + 2x_E \geq 30 - M(1 - z_3) \quad 4x_A + 3x_B + 5x_C - 2x_D + 3x_E \geq 50 \\
& 0x_A + 0x_B - 4x_C + 2x_D + 2x_E \geq 10 \\
& z_3 \leq z_1 \\
& z_3 \leq z_2 \\
& x_A, x_B, x_C, x_D, x_E \geq 0 \\
& x_A, x_B, x_C, x_D, x_E \text{ integers}, \\
& z_i \in \{0, 1\}, \quad \forall i \in \{1, 2, 3, \}.
\end{array}$$

1.4 Piecewise linear objective function

Consider again our original problem in Example 4. Suppose that in the problem, the objective function is a profit associated to five different products: A, B, C, D, and E. Instead of obtaining a profit of \$3 for each unit of product A, suppose that the profit for product A is

- \$3 per unit if $0 \leq x_A \leq 4$,
- \$4 per unit if $5 \leq x_A \leq 10$, and
- \$2 per unit if $11 \leq x_A$.

Assume that everything else remain the same.
Formulation:

- New binary decision variables: w_1, w_2, w_3. We will interpret them as follows:
  - $w_1 = 1$ means that the number of units of product A that is produced is from 0 to 4
  - $w_2 = 1$ means that the number of units of product A that is produced is from 5 to 10
  - $w_3 = 1$ means that the number of units of product A that is produced is 11 or greater

  So, we must add the constraint that exactly one of them take value one:

  $$w_1 + w_2 + w_3 = 1.$$

- New decision variables: x_{A1}, x_{A2}, x_{A3} which can take nonnegative integers values.
  - x_{A1} = the number of units of product A that is produced, whose value is from 0 to 4,
  - x_{A2} = the number of units of product A that is produced, whose value is from 5 to 10, or 0.
  - x_{A3} = the number of units of product A that is produced, whose value 11 or greater, or 0.

  For example, if $x_A = 6$, then $x_{A1} = 0, x_{A2} = 6, x_{A3} = 0$.

  We must add the following constraints to express the condition above:

  $$\begin{aligned} x_{A1} + x_{A2} + x_{A3} &= x_A, \\ 0 \le x_{A1} &\le 4w_1, \\ 4w_2 \le x_{A2} &\le 10w_2, \\ 11w_3 \le x_{A2} &\le Mw_3, \end{aligned}$$

  where M is a sufficiently large positive number.

- Then, we modify the objective function:

  $$\max(3x_{A1} + 4x_{A2} + 2x_{A3}) + 2x_B + 1x_C + 2x_D + 3x_E.$$

Hence, the new integer program is:

$$\begin{array}{rc} \max & (3x_{A1} + 4x_{A2} + 2x_{A3}) + 2x_B + 1x_C + 2x_D + 3x_E \\ s.t. & 4x_A + 2x_B + 5x_C + 3x_D + 2x_E \le 35 \\ & 2x_A + 4x_B + 3x_C + 4x_D + 1x_E \le 40 \\ & 1x_A + 1x_B + 3x_C + 2x_D + 2x_E \le 30 \\ & 4x_A - 3x_B + 5x_C - 2x_D + 3x_E \ge 50 \\ & 0x_A + 0x_B - 4x_C + 2x_D + 2x_E \ge 10 \\ & w_1 + w_2 + w_3 = 1 \\ & x_{A1} + x_{A2} + x_{A3} = x_A \\ & 0 \le x_{A1} \le 4w_1 \\ & 4w_2 \le x_{A2} \le 10w_2 \\ & 11w_3 \le x_{A2} \le Mw_3 \\ & x_A, x_{A1}, x_{A2}, x_{A3}, x_B, x_C, x_D, x_E \ge 0 \\ & x_A, x_{A1}, x_{A2}, x_{A3}, x_B, x_C, x_D, x_E \text{ integers} \\ & w_1, w_2, w_3 \in \{0, 1\}. \end{array}$$