ORIE 3310/5310
Lecture 16

14 March 2013.

Today:
- Stochastic Inventory Planning Problem.

Recall from last time:

- A betting game:
  - 3 plays
  - starts with 3 coins
  - ~~At~~ each play, if start with s coins can bet $x \in \{0, 1, \dots, \min(s, 5-s)\}$
    - if "wins" the play, gains ~~x~~ coins : $s+x$ total
    - if "loses" the play, loses x coins : $s-x$ total.
  - Wins a play w/ prob $2/3$
    loses a play w/ prob $1/3$.
  - Wins the game if at the end of the 3rd round, has 5 coins.
  - Goal: Max the prob of winning the game.
- DP Approach:
  1) Stage $\leftrightarrow$ Play
  2) State $\leftrightarrow$ # coins at start of the stage / play. $s \in \{0, 1, \dots, 5\}$
  3) Decision at stage k, state s:
     x_k = how many to bet at play #k.
     $x_k \in \{0, 1, \dots, \min(5, s-5)\}$.
  4) Opt. function
     $f_k^*(s)$ = max prob of winning the game if has s coins at start of stage k.
  5) Boundary conditions:
     $f_3^*(s)$ = max prob. of winning the game if has s coins at start of stage 3
     = max prob of having 5 coins after 1 play if currently has s coins.
  6) Recurrence relation:

$$f_k^*(s) = \max_{x_k = 0, \dots, \min(s, 5-s)} \left\{ \frac{1}{3} f_k^*(s - x_k) + \frac{2}{3} f_k^*(s + x_k) \right\}.$$

where if we start with S at stage k, then bet x_k:

$$\underbrace{\tfrac{1}{3} f_k^*(S - x_k)}_{\substack{\text{prob that we lose}\\ \text{the play but}\\ \text{win the game}}} + \underbrace{\tfrac{2}{3} f_k^*(S + x_k)}_{\substack{\text{prob that we win}\\ \text{the play and then}\\ \text{win the game}}} = \text{the prob that we win the game}$$

- Next, we consider a more interesting and "realistic" stochastic DP problem: An inventory planning problem, where the demands are not known deterministically ahead of time.

Ex: The problem:
- $T = 6$ periods.
- Costs: - no holding cost: $h_k = 0$.
  - cost per unit of production in period k: c_k
  - fixed cost
  - penalty per unit: $p = 72$ $\forall k = 1, \dots, T$.
- In each period: can produce at most 3 units.
  - can store at most 4 units in inventory.
- Demand in each period k = d_k is a random variable.

Period (k)	Cost c_k	Demand Probabilities $d_k = 0$	$d_k = 1$	$d_k = 2$	$d_k = 3$
1	54	0.3	0.3	0.4	0
2	56	0.2	0.3	0.4	0.1
3	58	0.1	0.2	0.4	0.3
4	57	0.1	0.2	0.4	0.3
5	50	0	0.2	0.3	0.5
6	48	0	0.2.	0.3	0.5

Remark:
- In this model, we are allowed to fall short on meeting demand, but everytime this happens, we will have to pay a penalty (a "lost sales penalty") per unit of item that we are short. We don't have to make this up in the next period

- The DP formulation :

1) Stage $k \leftrightarrow$ period k
2) At stage k

state $I \leftrightarrow$ inventory level I at the <u>start</u> of period k.

$I \in \{0,1,2,3,4\}$.

3) Decisions at stage k, state I :

x_k = # to produce at stage k.

where

$x_k \in \{0,1,2,3\}$.

<u>Note</u> : Suppose that $I + x_k > 4$ and d_k turns out to be 0.
Then we have more than we are allowed to store in our inventory.
→ In this case we keep 4 and dispose of the rest.

4) Opt function :

$f_k^*(I)$ = min <u>expected</u> total cost for periods $k, k+1, \ldots, T$.

5) Boundary conditions :

$f_{T+1}^*(I) = 0 \quad \forall\, I \in \{0,1,2,3,4\}$, <u>or</u> .

$$f_T^*(I) = \min_{x_T \in \{0,1,2,3\}} \left\{ x_T C_T + 1_{(x_k > 0)} F + \text{expected penalty} \right\}$$

$$= \min_{x_T \in \{0,1,2,3\}} \left\{ x_T C_T + 1_{(x_T > 0)} F + \sum_{j=0}^{3} P_{Tj} \cdot \max(0, d_T - (x_T + I)) P \right\}.$$

6) Recurrence relation:

$$f_K^*(I) = \min_{x_K \in \{0,1,2,3\}} \left\{ x_K c_K + 1_{(x_K>0)} F + \sum_{j=0}^{3} P_{Kj} \max(0, d_K - (x_K + I))P \right.$$
$$\left. + \sum_{j=0}^{3} P_{Kj} f_{K+1}^* \left(\max(\min(I + x_K - d_K, 4), 0) \right) \right\}$$

P_{Kj} = Probability that $d_K = j$.

<u>Why?</u>

* Suppose at stage K, state I, and suppose we produce x_K.

Then:

- Cost at ~~s~~tage k:
  1) production : $C_K X_K$
  2) fixed cost : $1_{(x_K>0)} F = \begin{cases} F & \text{if } x_K > 0 \\ 0 & \text{else} \end{cases}$.
  3) penalty cost: if $d_K > I + x_K$, then penalty is $= P \cdot (I + x_K - d_K)$
     if $d_K \leq I + x_K$, then penalty is $= 0$.
     → but demand is $d_K = j$ with probability P_{Kj}.

     So, <u>expected</u> penalty
     $$= \sum_{j=0}^{3} P(\text{demand} = j) \cdot (\text{penalty if demand} = j)$$
     $$= \sum_{j=0}^{3} P_{Kj} \cdot P \cdot \max(0, d_K - (I + x_K)).$$

- Cost at stages $k+1, \dots, T$:
  f_{k+1}^* (leftover)

But "leftover" quantity depends on d_K.

So, expected cost at stages $k+1, \ldots, T$ is :

$$\sum_{j=0}^{3} P(\text{demand}=j) \cdot f_{k+1}^{*}(\text{leftover if demand is } j)$$

$$= \sum_{j=0}^{3} P_{kj}\, f_{k+1}^{*}(\text{leftover is demand is } j)$$

where
"leftover if demand is j"
is $\min(I + x_k - d_k, 4)$ if $I + x_k > d_k$

$*$ zero if $I + x_k < d_k$.

$\therefore$ "leftover is demand is j" $= \max(0, \min(I + x_k - d_k, 4))$.

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7) Computation.