3 April 2013

Lecture 20

- Recall from last time:

  - Given an integer program (IP), we use the Gomory cutting planes approach to find an optimal solution:

    0) Solve an initial LP relaxation of (IP), call it (LP1)

    1) At the k^{th} iteration:
       solve (LPK).
       Let x^*_{LPK}, z^*_{LPK} denote the optimal (basic) solution and optimal value of (LPK).

    2) If $x^{(*)}_{LPK}$ is integer-valued, we're done. x^*_{LPK} is optimal for ~~LPK~~ (IP)

    Otherwise, $x^*_{LPK} = (x_1, \dots, x_n)$ is not all integers. So, there is i s.t. x_i is not an integer.

    Add the constraint:

if x_i is the r^{th} basic variable in the tableau. →

$$x_i + \sum_{x_j \text{ nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_i \leq \lfloor \bar{b}_r \rfloor$$

to (LPK). Call this new linear program (LPK+1).

Repeat Step 1.

$$\therefore \quad x_i + \overbrace{\sum_{j\ \text{nonbasic}} \bar{a}_{rj} x_j}^{\text{zero}} = \bar{b}_r$$

$$x_i = \bar{b}_r > \lfloor \bar{b}_r \rfloor$$

$$\text{So,} \quad x_i \not\leq \lfloor \bar{b}_r \rfloor$$

$$\therefore \quad x_i + \sum_{j\ \text{nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_j \not\leq \lfloor \bar{b}_r \rfloor$$

$\therefore$ x^*_{LPK} does not satisfy $\circledast$.

Next, we prove ①:

We know that all (IP) solutions satisfy

$$x_i + \sum_{j\ \text{nonbasic}} \bar{a}_{rj} x_j \leq \bar{b}_r .$$

We also know that

$$x_i + \sum_{j\ \text{nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_j \leq x_i + \sum_{j\ \text{nonbasic}} \bar{a}_{rj} x_j .$$

$$\leq \bar{b}_r$$

$\therefore$ If x is an (IP) feasible solution, it satisfies

$$x_i + \sum_{j\ \text{nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_j \leq \bar{b}_r = \lfloor \bar{b} \rfloor + \varepsilon$$

where $0 < \varepsilon < 1$

Also, if x is (IP) feasible solution, then

$x_i + \sum_{j\ \text{nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_j$ is integer-valued

$$\text{So,} \quad x_i + \sum_{j\ \text{nonbasic}} \lfloor \bar{a}_{rj} \rfloor x_j \leq \lfloor \bar{b} \rfloor . \quad \checkmark$$

$\square$.

$\therefore$ ① is satisfied.

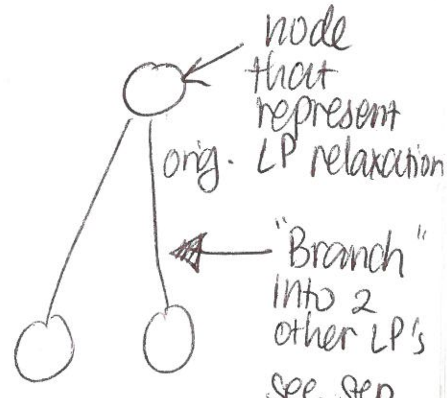
Branch and Bound

- Branch and Bound is another method for solving integer programs by solving a sequence of linear programs related to it.

- We first outline the method, then discuss the idea and the similarities & differences with cutting plane methods.

- Here is the method:

  ⓪ Take original LP relaxation
  - Maintain a "tree" that keeps track of the LP's solved. We start by having a node corresponding to the original LP relaxation.
  - Let $z_0 = -\infty$ (for a maximization problem).



  ① Choose an unfathomed branch, solve the LP. if infeasible, do not branch (i.e. we have "fathomed" the branch).
  Suppose x_{LP}^* is optimal for this LP.

  ⓐ If x_{LP}^* is integer-valued, with the corresponding objective function value: $z_{LP} := c^T x_{LP}^*$, then:

  - Check: if $z_{LP} \le z_0$, do nothing to z_0.

  - If $z_{LP} > z_0$, update the value of z_0:
  $$z_0 = z_{LP},$$

  We <u>do not branch</u> since the solution is integral (we <u>have "fathomed"</u> this branch and don't need to branch further). Go to step ②.

and $z_{LP} := c^T x_{LP}^* \geq z_0$

(b) If x_{LP}^* is not integer-valued ✓, choose x_j where $x_j = \alpha_j$ is not an integer.

Then, we have 2 LP's (to solve in later iterations):

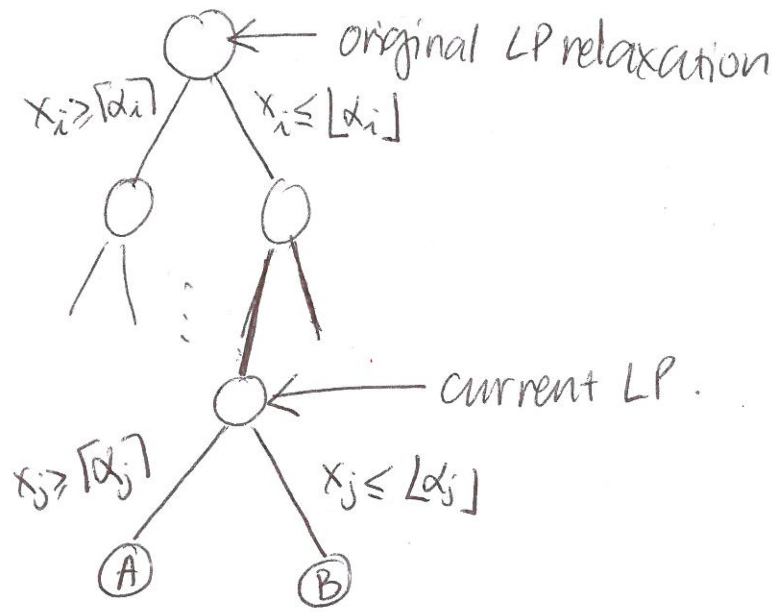
(A): Current LP, with additional constraint:

$$x_j \geq \lceil \alpha_j \rceil$$

(B) Current LP with additional constraint:

$$x_j \leq \lfloor \alpha_j \rfloor$$

Update the tree by "<u>branching</u>" from the current LP node, then go to step (2)



(c) If x_{LP}^* is not integer-valued, and $z_{LP} = c^T x_{LP}^* \leq z_0$, then we <u>do not branch</u> (we have "<u>fathomed</u>" the current branch and don't need to branch further). Then, go to step (2).

(2) Pick another LP from one of the nodes in the tree that has not been fathomed. Then go to step (1),

until all branches have been fathomed, then stop.
Go to step ③.

③ If an integer-valued solution exists, z_0 is finite.
The IP solution ~~corresponds~~ is the integer-valued solution that corresponds to the value of z_0.

It is possible that z_0 is $-\infty$ in which case no integer-valued solution was found.

- Main idea:
  - After solving each LP, 2 things can happen:
    ① We <u>branch</u>: We have 2 other LP's to solve. The 2 new LP's are the same as the current LP, with one additional constraint:

$$x_j \le \lfloor \alpha_j \rfloor \quad \text{for one LP, and}$$

$$x_j \ge \lceil \alpha_j \rceil \quad \text{for the other}$$

where α_j is the value of the basic variable x_j at current LP solution and α_j is not an integer

    ② We fathom the current node, and <u>do not branch</u>.
    There are 3 cases for which we do not branch:

    ① Current LP is infeasible
    ② Current LP has integer optimal solution
    ③ The optimal value of the current LP, call it z_{LP}, is smaller than z_0,
    where z_0 = best objective function value among all integer-valued solutions found so far.

"worse" assuming it's a <u>max</u> problem

- Similarity with cutting plane method:
  - We solve linear programs until the optimal solution to the IP is found (or until we show that the IP is infeasible)

- Differences with cutting plane:

(Gomory) Cutting Plane	Branch and Bound
• Generate one new LP at a time	• Generate zero or two LP's at a time (fathom → 0 LP, branch → 2 LP's).
• Stop when an integer-valued solution is found. This solution is optimal for the IP.	• When an integer-valued soluti[illegible] is found, move on to another LP/branch, since a better integer solution might be found. • Stop only when all branches have been fathomed.

- Note: the ~~TSP~~ method for solving TSP from last week is a cutting plane method (~~but~~ we generate the constraints not the Gomory way, but by identifying violated constraints of the form:

"subtour elimination constraints" $\longrightarrow$ $$\sum_{\substack{i \in S \\ j \notin S}} x_{ij} \geq 2 \qquad S \subseteq V = \{1, 2, \ldots, n\}.$$

Example

$$\text{IP}: \left\{\begin{array}{ll} \text{Max} & 3x_1 - x_2 \\ \text{s.t.} & 3x_1 - 2x_2 \le 3 \\ & -5x_1 - 4x_2 \le -10 \\ & 2x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0 \\ & x_1, x_2 \text{ integers} \end{array}\right.$$

Original LP relaxation:

$$\begin{array}{ll} \text{Max} & 3x_1 - x_2 \\ \text{s.t.} & 3x_1 - 2x_2 \le 3 \\ & -5x_1 - 4x_2 \le -10 \\ & 2x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0. \end{array}$$

Using Gomory cutting planes method:

Step ⓪: Solve original LP relaxation.

Optimal tableau:

	x_1	x_2	x_3	x_4	x_5	RHS
$(-z)$	0	0	$-5/7$	0	$-3/7$	$-30/7$
x_1	1	0	$1/7$	0	$2/7$	$13/7$ ←
x_4	0	0	$-3/7$	1	$22/7$	$31/7$
x_2	0	1	$-2/7$	0	$3/7$	$9/7$.

Step ①: First iteration.

$x_1 = 13/7$, $x_2 = \overset{9}{\cancel{31}}/7$ $\therefore$ not integers.

Choose row 1: $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_5 = \frac{13}{7}$.

Gomory cutting plane:

$$\left(\frac{1}{7} + \left\lfloor\frac{1}{7}\right\rfloor\right)x_3 + \left(\frac{2}{7} - \left\lfloor\frac{2}{7}\right\rfloor\right)x_5 \ge \frac{13}{7} - \left\lfloor\frac{13}{7}\right\rfloor$$

$\therefore \frac{1}{7}x_3 + \frac{2}{7}x_5 - S_1 = \frac{6}{7}$ (~~add~~ subtract surplus var)

$x_1 + s_1 = 1 \Leftrightarrow x_1 \le 1$.

or $\quad -\frac{1}{7}x_3 - \frac{2}{7}x_5 + s_1 = -\frac{6}{7}$. $\quad -\frac{1}{7}x$

$-x_3 - 2x_5 + s_1$

Add the constraint:

	x_1	x_2	x_3	x_4	x_5	s_1	RHS
$(-z)$	0	0	$-5/7$	0	$-3/7$	0	$-30/7$
x_1	1	0	$1/7$	0	$2/7$	0	$13/7$
x_4	0	0	$-3/7$	1	$22/7$	0	$31/7$
x_2	0	1	$-2/7$	0	$3/7$	0	$9/7$
s_1	0	0	$-1/7$	0	$-2/7$	1	$-6/7$

negative. $\therefore$ use dual simplex to reoptimize

	x_1	x_2	x_3	x_4	x_5	s_1	RHS
$(-z)$	0	0	$-1/2$	0	0	$-3/2$	-3
x_1	1	0	0	0	0	1	1
x_2	0	1	$-1/2$	0	0	$3/2$	0
x_4	0	0	-2	1	0	11	-5
x_5	0	0	$1/2$	0	1	$-7/2$	3

negative $\therefore$ use dual simplex

	x_1	x_2	x_3	x_4	x_5	s_1	RHS
$(-z)$	0	0	0	$-1/4$	0	$-17/4$	$-7/4$
x_1	1	0	0	0	0	1	1
x_2	0	1	0	$-1/4$	0	$-5/4$	$5/4$
x_3	0	0	1	$-1/2$	0	$-11/2$	$5/2$
x_5	0	0	0	$1/4$	1	$-3/2$	$7/4$

optimal tableau for the new LP but still not integer-valued.

Iteration 2:

Choose the $(-z)$ row to get a cut:

$$-z - \frac{1}{4}x_4 - \frac{17}{4}s_1 = -\frac{7}{4}$$

or $$z + \frac{1}{4}x_4 + \frac{17}{4}s_1 = \frac{7}{4}.$$

So, the cut:

$$\left(\frac{1}{4} - \left\lfloor\frac{1}{4}\right\rfloor\right)x_4 + \left(\frac{17}{4} - \left\lfloor\frac{17}{4}\right\rfloor\right)s_1 \geq \frac{7}{4} - \left\lfloor\frac{7}{4}\right\rfloor$$

$$\therefore \quad \frac{1}{4}x_4 + \frac{1}{4}S_1 - S_2 = \frac{3}{4}$$

← (subtract surplus variable S_2)

$$\text{or} \quad -\frac{1}{4}x_4 - \frac{1}{4}S_1 + S_2 = -3/4$$

Add the constraint:

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	RHS
$(-z)$	0	0	0	-1/4	0	-17/4	0	-7/4
x_1	1	0	0	0	0	1	0	1
x_2	0	1	0	-1/4	0	-5/4	0	5/4
x_3	0	0	1	-1/2	0	-11/2	0	5/2
x_5	0	0	0	1/4	1	-3/4	0	7/4
S_2	0	0	0	-1/4	0	-1/4	1	-3/4

← negative ∴ use dual simplex

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	RHS
$-z$	0	0	0	0	0	-4	-1	-1
x_1	1	0	0	0	0	1	0	1
x_2	0	1	0	0	0	-1	-1	2
x_3	0	0	1	0	0	-5	-2	4
x_4	0	0	0	1	0	1	-4	3
x_5	0	0	0	0	1	-1	1	1

} $x_1 = 1$, $x_2 = 2$ ∴ optimal for IP :)

$$\frac{1}{4}x_4 + \frac{1}{4}S_1 - S_2 = \frac{3}{4} \leftrightarrow x_1 + x_2 \geq 3.$$

Using Branch and Bound

Step (0) Solve LP relaxation: $\underbrace{x_1 = 13/7,\ x_2 = 9/7}_{x_{LP}^*}$, ← see soln from previous part
with optimal value $z_{LP} = 30/7$.

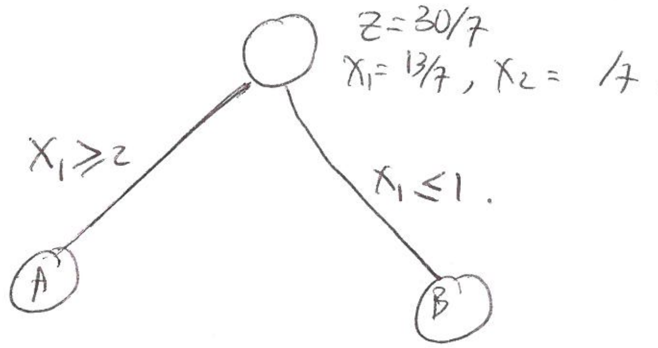
Iteration 1

Step (1) Since x_{LP}^* is not integer-valued, and $z_{LP} > z_0 (= -\infty)$, then branch with additional constraint:

$$(A):\ x_1 \geq \lceil 13/7 \rceil = 2$$

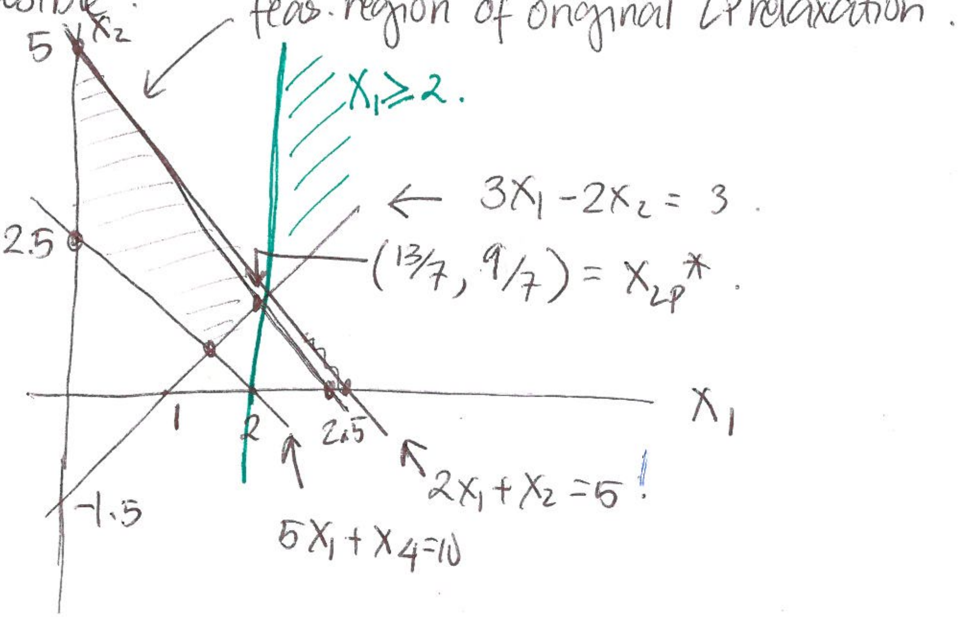
$$(B):\ x_1 \leq \lfloor 13/7 \rfloor = 1$$

Current tree:



~~Step (2)~~ Iteration 2: Consider node A.

By graphing the feasible region, we see that A is infeasible:



So, we fathom node A, by infeasibility.

<u>Iteration 3</u> The only other branch not yet explored or fathomed is node B.

We solve

$$
\begin{aligned}
\text{Max } & 3x_1 - x_2 \\
\text{s.t. } & \left.\begin{aligned} 3x_1 - 2x_2 &\le 3 \\ -5x_1 - 4x_2 &\le -10 \\ 2x_1 + x_2 &\le 5 \end{aligned}\right\} \text{original constraints} \\
& x_1 \le 1 \quad \leftarrow \text{new} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

The opt solution is: $x_{LP}^* = (1, 5/4)$, with

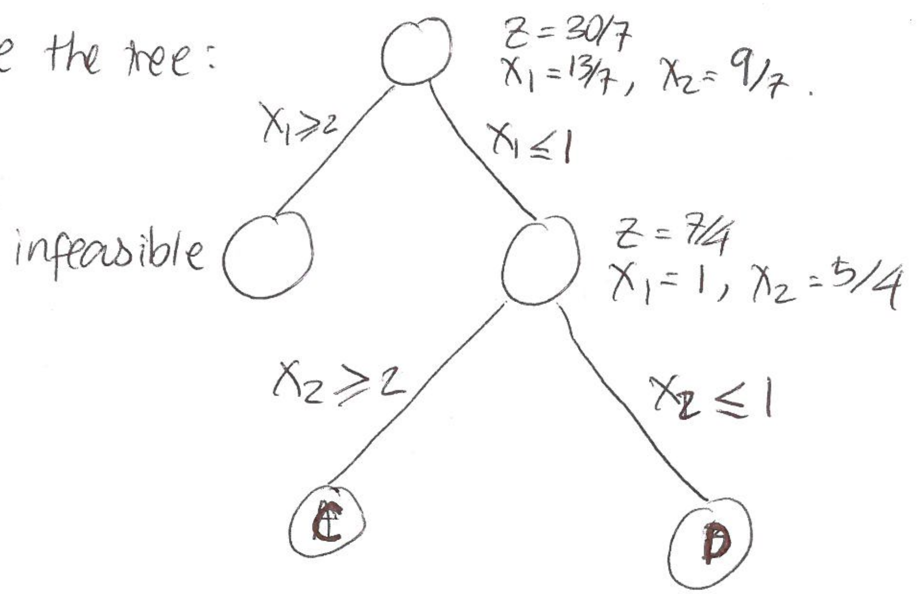
$z_{LP} = 7/4$.

Since x_{LP}^* is not integer-valued and $z_{LP} > z_0 (= -\infty)$, then <u>branch</u> with the additional constraint:

C : $x_2 \ge \lceil 5/4 \rceil = 2$

D : $x_2 \le \lfloor 5/4 \rfloor = 1$

Update the tree:



Iteration 4 : Consider node C.

We solve: Max $3x_1 - x_2$

$$
\begin{aligned}
\text{s.t.}\quad & 3x_1 - 2x_2 \le 3 \\
& -5x_1 - 4x_2 \le -10 \\
& 2x_1 + x_2 \le 5
\end{aligned}
\Bigg\}\ \text{original}
$$

$x_1 \le 1$ ← added

$x_2 \ge 2$ ← added

$x_1, x_2 \ge 0$.

The opt solution is: $\underbrace{x_1 = 1,\ x_2 = 2}_{x_{LP}^* = (1,2)}$, $z_{LP} = 1$

Since $z_0 < z_{LP}$, update z_0:

$z_0 = 1 (= z_{LP})$

Since x_{LP}^* has integer values, ∴ fathom this branch (∴ don't branch further). Update the tree

Iteration 5 : Consider node D.

We solve: Max $3x_1 - x_2$

$$
\begin{aligned}
\text{s.t.}\quad & 3x_1 - 2x_2 \le 3 \\
& -5x_1 - 4x_2 \le -10 \\
& 2x_1 + x_2 \le 5
\end{aligned}
\Bigg\}\ \text{original}
$$

$x_1 \le 1$ ← added

x_2 ← added

$x_1, x_2 \ge 0$.

We can see graphically that this LP is infeasible.

∴ fathom this branch.

Note that all branches have now been fathomed.

$z_0 = 1$, corresponding to the integer solution $x_1 = 1, x_2 = 2$.

This is the opt soln to the LP.

The final tree: