ORIE 3310/5310/5311
Lecture 10

Solving Inventory Planning Problems using DP

- The following is an example of a "basic" inventory planning problem.

- There are other Inventory planning models that are more complicated and have more components, but this example should illustrate the basic ideas of how DP is used to solve inventory planning problems in general.

- In a basic inventory planning problem,
  - We are given the following input:
    - There are T periods
    - For each period, there is a demand d_i, $i = 1, 2, \ldots, T$
    - A production cost per unit in each period, c_i.
    - A holding cost per unit in each period, h_i
    - (A fixed-cost in each period, f_i.)
  - Decision: to determine quantity to be produced in each period
  - Constraint: subject to satisfying all demand in each period
  - Objective: To minimize total cost.
    = total over T periods of production, holding, and fixed costs.

  - Some notes:
    - * A fixed-cost f_i is charged if any positive quantity is produced in period i.
    - * A holding cost h_i is charged for each unit that is stored in the inventory between periods $i-1$ and i
    - * Assume that production is done at the middle of each period i. At the end of period $i-1$, demand d_{i-1} is satisfied, and leftover items are stored in the inventory, charged h_i dollars per unit.

Ex: $T = 4$ months.

Period (i)	Demand (d_i)	Production Cost/unit (c_i)	Holding cost/unit (h_i)	Fixed Cost (f_i)
1	10	3	0.2	5
2	40	2	0.3	20
3	20	4	0.5	10
4	50	3	0.8	10

Solving this problem using DP

1) Specify the stages

stage $k \Leftrightarrow$ period k. ← quite a natural correspondence!

2) Specify the states at each stage.

- This step is incredibly important, yet very tricky!
- To gain some intuition to what "states" mean, and how to identify what the states should be in various problem:

→ Start by thinking what kind of decisions we are supposed to make at each stage.

→ Imagine making a series of these decisions in the previous $k-1$ stages, then these decisions influence the current state that we are at.

→ Ex: In the shortest-path problem, at the start, we are at node s, then we make a decision of which edge to take, which influenced at which node we would be at in the next stage.

So, states $\Leftrightarrow$ nodes.

→ In this problem, we decide how many to produce each month. Based on this decision and the demand, we would have a particular number of items in the inventory.

So, we can have <u>inventory levels</u> as <u>states</u>.

In stage 1: Possible inventory levels: $\{0\}$

Stage 2: $\{0, 10, 20, \ldots, 40+20+50=110\}$.

Stage 3: $\{0, 10, 20, \ldots, 20+50=70\}$.

Stage 4: $\{0, 10, 20, \ldots, 50\}$.

Dummy stage → Stage 5: $\{0\}$

3) Specify the set of possible decisions for each state I, at each stage k: call it $Q_{I,k}$. [Assume production must be multiples of 10].

→ At each stage we decide how much to produce, depending on our current state.

<u>EX</u>: In stage 1, we start with no inventory, and a demand of 10.
So, we cannot produce fewer than 10 units, but we can produce more.
We observe, though, that we probably don't have to produce more than 120 units (total demand).

<u>Ex</u>: In stage 3, we have a demand of 20.
What is our inventory level? ← there are a few possible states.
For each possible inventory level: 0, 10, 20, 30, ...
we determine the allowable decisions.

If the inventory level is 0, must produce at least 20, but no more than 70.
If the inventory level is 10, must produce at least 10, but no more than 60
etc.

So:

$$Q_{0,1} = \{10, 20, 30, \ldots, 120\}.$$

$$Q_{I,2} = \{\max\{40-I, 0\}, \ldots, 110-I\}$$

$$Q_{I,3} = \{\max\{20-I, 0\}, \ldots, 70-I\}$$

$$Q_{I,4} = \{\max\{50-I, 0\}, \ldots, 50-I\} = \{50-I\}.$$

$$\therefore Q_{I,K} = \{\max\{d_k - I, 0\}, \ldots, \sum_{i=K}^{T} d_i\}$$

4) Describe the optimization function to be solved at state I and stage k.
(in words):

At inventory level I in period k:

dummy period

let: $f_k^*(I)$ = the minimum total remaining cost to meet demands in period k until period $T+1$

5) Specify "boundary conditions" (i.e. the "smallest subproblem"):

$f_{T+1}^*(I)$ = the minimum total remaining cost to meet demands in period $T+1$ until period $T+1$

just 1 period!

$f_{T+1}^*(I) = 0$ for all I.

6) Write the recurrence relation for $f_k^*(I)$

→ i.e. write $f_k^*(I)$ in terms of $f_{k-1}^*(\cdot)$ or $f_{k+1}^*(\cdot)$

→ Since our boundary condition in part (5) is the value in the last period, then we'll write $f_k^*(I)$ in terms of $f_{k+1}^*(\cdot)$.

$$f_k^*(I) = \min_{x_k \in Q_{I,k}} \Big\{ \underbrace{x_k c_k + \mathbb{1}_{\{x_k>0\}} f_k + (I+x_k-d_k) h_k}_{\text{total cost due to production and holding costs incurred in period } k.} + \underbrace{f_{k+1}^*(I+x_k-d_k)}_{\text{the min. total cost to meet demands in periods } k+1,\dots,T.} \Big\}$$

7) Compute Backwards via recurrence relation, starting from the boundary conditions in step (5):

Stage $T+1=5$ $\quad f^*_{T+1}(0)=0.$

Stage 4 : $f_4^*(0) = \min\limits_{x_4 \in Q_{I,k}} \left\{ x_k \cdot 3 + \mathbb{1}_{\{x_4>0\}} \cdot 10 + (I+x_k-50)\cdot 0.8 + f_5^*(I+x_k-50) \right\}$

where $Q_{I,k}$ = possible decisions
$= \{\max\{50-0, 0\}, \dots, 50\}$
$= \{50\}.$

$\therefore f_4^*(0) = 50\cdot 3 + 10 + 0 + \underbrace{f_5^*(0)}_{0} = 160 \quad \boxed{x_4=50}$

$f_4^*(10) = \min\limits_{x_4 \in \{40\}} \left\{ 3x_k + \mathbb{1}_{\{x_k>0\}} 10 + (I+x_k-50)\cdot 0.8 + f_5^*(I-x_k-50) \right\}$

$= 120+10+0+f_5^*(0) = 130, \quad \boxed{x_4=40}$

$f_4^*(20) = \min\limits_{x_4 \in \{30\}} \left\{ 3x_k + \mathbb{1}_{\{x_k>0\}} 10 + (I+x_k-50)\cdot 0.8 + f_5^*(I+x_k-50) \right\}$

$= 90+10+0+f_5^*(0) = 100, \quad \boxed{x_4=30}.$

$f_4^*(30) = \min\limits_{x_4 \in \{20\}} \{ \dots \} = 60+10+0+f_5^*(0) = 70, \quad \boxed{x_4=20}.$

$f_4^*(40) = \min\limits_{x_4 \in \{10\}} \{ \dots \} = 30+10+0+f_5^*(0) = 40, \quad \boxed{x_4=10}$

$f_4^*(50) = \min\limits_{x_4 \in \{0\}} \{ \dots \} = 0+0+0+f_5^*(0) = 0, \quad \boxed{x_4=0}.$

<u>Stage 3</u> : $f_3^*(0) = \min\limits_{x_3 \in \{20,\dots,70\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (0 + x_3 - 20)\cdot 0.5 + f_4^*(0 + x_3 - 20) \right\}$

$$= \min \{ 80 + 10 + 0 + f_4^*(0), \quad \leftarrow x_3 = 20$$

$$120 + 10 + 5 + f_4^*(10), \quad \leftarrow x_3 = 30$$

$$160 + 10 + 10 + f_4^*(20), \quad \leftarrow x_3 = 40$$

$$200 + 10 + 15 + f_4^*(30), \quad \leftarrow x_3 = 50$$

$$240 + 10 + 20 + f_4^*(40), \quad \leftarrow x_3 = 60$$

$$280 + 10 + 25 + f_4^*(50) \}.$$

$$= \min \{ 90 + 160, 135 + 130, 180 + 100, 225 + 70, 270 + 40, 315 + 0 \}$$

$$= 250, \quad \boxed{x_3 = 20}.$$

$$f_3^*(10) = \min_{x_3 \in \{10,\dots,60\}} \left\{ 4x_3 + \mathbb{1}_{x_3>0} \cdot 10 + (10 + x_3 - 20)\cdot 0.5 + f_4^*(10 + x_3 - 20) \right\}$$

$$= \min \{ 50 + f_4^*(0), 95 + f_4^*(10), 140 + f_4^*(20),$$

$$185 + f_4^*(30), 230 + f_4^*(40), 275 + f_4^*(50) \}$$

$$= \min \{ 50 + 160, 95 + 130, 140 + 100, 185 + 70, 230 + 40,$$

$$315 + 0 \} = 210, \quad \boxed{x_3 = 10}.$$

$$f_3^*(20) = \min_{x_3 \in \{0,\dots,50\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (20 + x_3 - 20)\cdot 0.5 + f_4^*(20 + x_3 - 20) \right\}$$

$$= \min \{ 0 + f_4^*(0), 55 + f_4^*(10), 100 + f_4^*(20),$$

$$145 + f_4^*(30), 190 + f_4^*(40), 235 + f_4^*(50) \}$$

$$= 160, \quad \boxed{x_3 = 0}.$$

$$f_3^*(30) = \min_{x_3 \in \{0,\dots,40\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (30 + x_3 - 20)\cdot 0.5 + f_4^*(30 + x_3 - 20) \right\}$$

$$= \min \{ 5 + f_4^*(10), 60 + f_4^*(20), 105 + f_4^*(30), 150 + f_4^*(40),$$

$$195 + f_4^*(50) \} = 135, \quad \boxed{x_3 = 0}.$$

$$f_3^*(40) = \min_{x_3 \in \{0,\dots,30\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (40 + x_3 - 20)\cdot 0.5 + f_4^*(40 + x_3 - 20) \right\}$$

$$= \min \{ 10 + f_4^*(20), 65 + f_4^*(30), 110 + f_4^*(40), 155 + f_4^*(50) \}$$

$$= 110, \quad \boxed{x_3 = 0}.$$

$$f_3^*(50) = \min_{x_3 \in \{0,10,20\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (50 + x_3 - 20) \cdot 0.5 + f_4^*(50 + x_3 - 20) \right\}$$

$$= \min\{ 15 + f_4^*(30),\ 70 + f_4^*(40),\ 115 + f_4^*(50) \}$$

$$= 85, \quad \boxed{x_3 = 0}.$$

$$f_3^*(60) = \min_{x_3 \in \{0,10\}} \left\{ 4x_3 + \mathbb{1}_{\{x_3>0\}} \cdot 10 + (60 + x_3 - 20) \cdot 0.5 + f_4^*(60 + x_3 - 20) \right\}$$

$$= \min\{ 20 + f_4^*(40),\ 75 + f_4^*(50) \}$$

$$= 60, \quad \boxed{x_3 = 0}.$$

$$f_3^*(70) = 25 + f_4^*(50) = 25, \quad \boxed{x_3 = 0}.$$

<u>Stage 2</u>: Compute $f_2^*(I)$ for $I \in \{0, 10, \ldots, 110\}$.

<u>Stage 1</u>: Compute $f_1^*(I)$ for $I = 0$