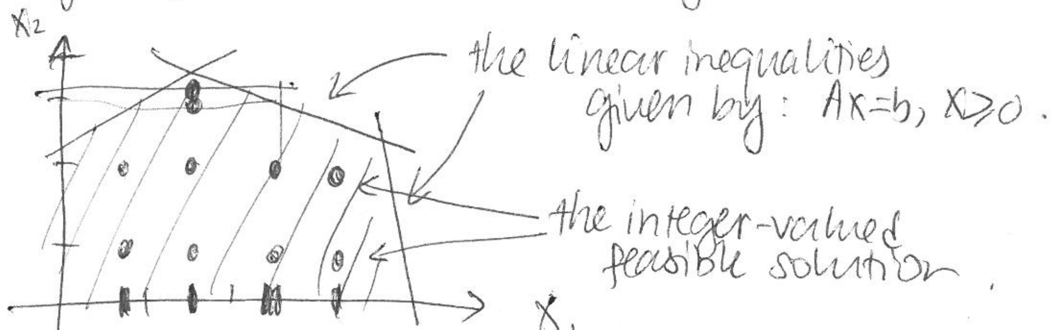
Lecture 19 2April2013.

Today:
- Gomory cutting plane.

~~Ex~~: • Suppose that we'd like to find an opt soln to the following integer program (Call it (IP)):

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & x \text{ integer} \end{aligned} \qquad (IP).$$

• The feasible region of (IP) will look something like:



the shaded region is the LP feasible region (if there hadn't been the constraint "x integer").

• The linear program that we obtain after removing the integrality constraint from (IP) is called <u>the LP-relaxation of (IP)</u>.

Let us denote this LP by (LP1):

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad (LP1).$$

Noe that

- The feasible region of (IP) is contained inside the feasible region of (LP1)

- This means that the optimal value of (LP1) is better than that of (IP) (because it has a larger feas region). In our example, since they are both maximization problems, then better means greater than:

$$\text{if } \quad z_{IP}^* = \text{opt value of (IP)}$$

$$z_{LP1}^* = \text{opt value of (LP1)}$$

$$\text{then} \quad z_{IP}^* \leq z_{LP1}^* . \qquad (*)$$

$\therefore$ z_{LP1}^* is an upper bound for z_{IP}^*.

- Suppose that x_{LP1}^* is a basic feasible solution that is optimal for LP1, with corresponding opt value z_{LP1}^*.

  If x_{LP1}^* is integer-valued, then it is feasible for (IP), and in fact it is optimal for (IP).

  Why?
  - Feasible because: x_{LP1}^* satisfy all linear constraints and it is integer-valued. $\therefore$ satisfy all (IP) constraints
  - Optimal because:
    1. We know from $(*)$ that $z_{IP}^* \leq z_{LP1}^*$.
    2. Since x_{LP1}^* is feasible for (IP), then

       $$z_{LP1}^* \leq z_{IP}^*$$

    3. So, from 1 & 2: $z_{LP1}^* = z_{IP}^*$ ✓.

- So, if we solve (LP1) and obtain an integer-valued solution, we're done!

  We say that (LP1) has an <u>integrality property</u>.

  <u>Ex / Recall</u>: Some examples of linear programs w/ integrality properties:

  - LP for mincost flow problem with integer-valued supply values and edge capacities.
  - LP for maxflow / mincut with integer-valued edge capacities.
  - LP for the assignment problem.

- But we need to develop a method for solving integer programs if their LP-relaxations do not have ~~the~~ integer-valued optimal solutions.

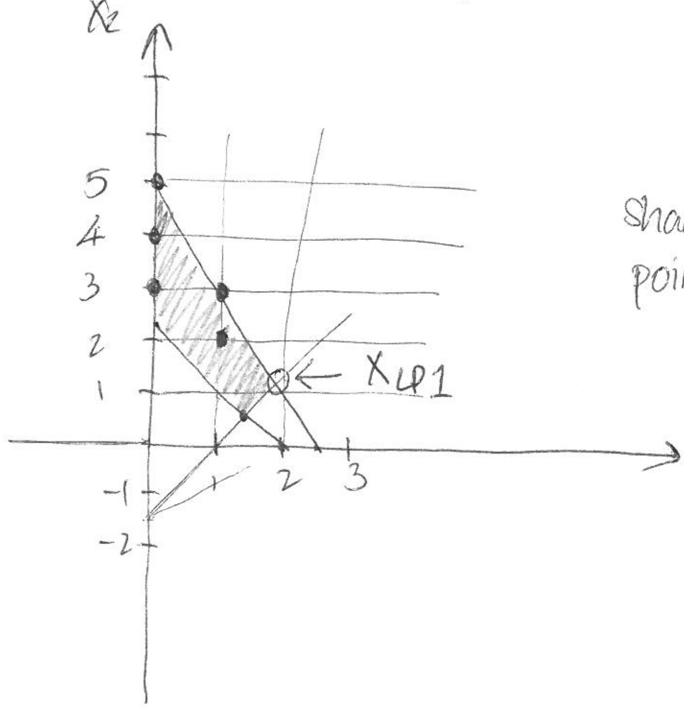
  <u>Ex</u>:

$$
\begin{aligned}
\text{Max } \; & 3x_1 - x_2 \\
\text{s.t. } \; & 3x_1 - 2x_2 \leq 3 \\
& -5x_1 - 4x_2 \leq -10 \qquad \text{(IP)} \\
& 2x_1 + x_2 \leq 5 \\
& x_1, x_2 \geq 0, \text{ integer}
\end{aligned}
$$

[Add slack variables to make equality constraints:

$$
\begin{aligned}
\text{Max } \; & 3x_1 - x_2 \\
\text{s.t. } \; & 3x_1 - 2x_2 + x_3 = 3 \\
& -5x_1 - 4x_2 + x_4 = -10 \\
& 2x_1 + x_2 + x_5 = 5 \\
& x_1, x_2, \ldots, x_5 \geq 0, \text{ integers}.
\end{aligned}
$$
]

A sketch of the feasible region:



Shaded = (LP1) feas region.
points = (IP) feas region.

Solving for (LP1):

$$
\begin{aligned}
\text{Max } & 3x_1 - x_2 \\
\text{s.t. } & 3x_1 - 2x_2 + x_3 = 3 \\
& -5x_1 - 4x_2 + x_4 = -10 \\
& 2x_1 + x_2 + x_5 = 5 \\
& x_1, \ldots, x_5 \geq 0
\end{aligned}
\qquad \text{(LP1)}
$$

We obtain an opt solution:

$x^*_{LP1} = (x_1, x_2) = (13/7, 9/7)$ not integer-valued!

$\left((x_1, x_2, x_3, x_4, x_5) = (13/7, 9/7, 0, 31/7, 0) \right)$.

with objective value $z^*_{LP1} = 30/7$.

<u>Remark</u>: We don't know what the opt (IP) solution is yet! However, we do know that

$$z^*_{IP} \leq 30/7.$$

Our goal:

1. To obtain an LP-relaxation of (IP) that has an integer-valued optimal (basic feasible) solution. This allows us to obtain a solution for (IP) itself.

2. To obtain an upperbound to (IP) (if (IP) is a maximization problem) that is ~~the bes~~ as good as possible.

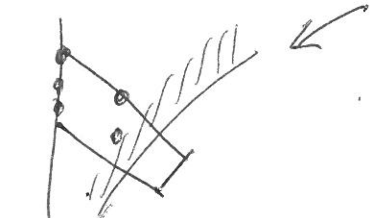
Given an integer programming problem (IP):

Defn A <u>cutting plane</u> is a constraint that is

1. Satisfied by all feasible solution to (IP)

2. "Removes" some ~~area~~ points from the LP-relaxation of (IP) i.e. not satisfied by these points.

<u>Ex</u>: In picture:



$\alpha_1 x_1 + \ldots + \alpha_n x_n \le \beta$ is a cutting plane.

<u>Remark</u>:

- The LP given by (LP1) plus the additional constraint

$$\alpha_1 x_1 + \ldots + \alpha_n x_n \le \beta$$

~~call this~~ gives us a new LP-relaxation of (IP). Call this (LP2).

If x^*_{LP2}, z^*_{LP2} denote the opt solution and the opt value of (LP2), respectively, then we know that

$$z^*_{IP} \le z^*_{LP2} \le z^*_{LP1}$$

Why?

- $z_{IP}^* \leq z_{LP2}^*$

  because the feas. region of IP is contained in the feas. region of LP2, and we have a maximization problem.

- $z_{LP2}^* \leq z_{LP1}^*$

  because the feas. region of IP is contained in the feas region of LP1, and we have a maximization problem.

This suggests the following method for solving integer programs:

0) First, solve the LP-relaxation of (IP). Call it (LP1).

1) In the k^{th} iteration, solve the current LP: (LPk). Let x_{LPk}^*, z_{LPk}^* denote its opt solution & value.

2) If x_{LPk}^* is integer-valued, we are done. It is optimal for (IP). So, $z_{IP}^* = z_{LPk}^*$.

   Otherwise, if x_{LPk}^* is not integer valued:
   - Add one (or more) cutting plane:

     $$\alpha_1 x_1 + \ldots + \alpha_n x_n \leq \beta$$

     to our current LP.

     Call the new LP (LP(k+1)).

   Repeat step 1.

There are various way for coming up with suitable cutting planes. However, Gomory came up with one method that can be used very generally:

Suppose x_{LPK}^* is not integer-valued,
then there is some i such that x_i is not integer.

Add the constraint:

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor$$

where:

- N = index set of nonbasic variables.
- If B = matrix of the columns of basic variables, then let
  $$\bar{A} := B^{-1} A$$
  $$\bar{b} := B^{-1} b$$
- $\bar{a}_{ij}$:= the (ij) entry of $\bar{A}$
  = the entry of $\bar{A}$ corresponding to row of x_i and the j^{th} column.
- $\bar{b}_i$ = the entry of $\bar{b}$ corresponding to the row of x_i.

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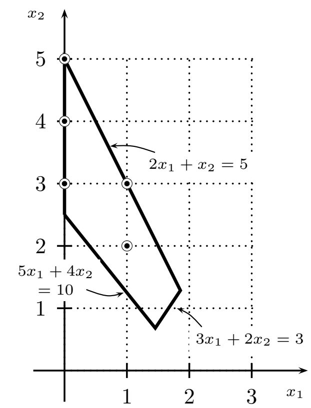
1 Example

Consider the following IP problem

$$\begin{array}{rrcrcl} \max\ z & = & 3x_1 & - & x_2 & \\ \text{s.t.} & & 3x_1 & - & 2x_2 & \leq 3 \\ & & -5x_1 & - & 4x_2 & \leq -10 \\ & & 2x_1 & + & x_2 & \leq 5 \\ & & & & x_1,\ x_2 & \geq 0, \quad \text{integer} \end{array}$$



The feasible (integer) points, denoted $\odot$, along with the feasible region of the corresponding LP relaxation are depicted on the figure to the right.

Iteration 1:

Solve LP1:We solve the first LP relaxation. It turns out that the optimal LP solution tableau reads

	x_1	x_2	x_3	x_4	x_5	RHS	
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	$z = \frac{30}{7}$	
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	$\frac{13}{7}$	$\leftarrow$ source row for cut
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	$\frac{9}{7}$	
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	$\frac{31}{7}$	

Add a Gomory cutting plane: The first row is the source row (the row that we'll use to get the cut):

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_5 = \frac{13}{7},$$

so, the cut is

$$x_1 + 0x_3 + 0x_5 = 1,$$

Adding slack variable $x_6 \geq 0$ yields

$$x_1 + x_6 = 1. \tag{1}$$

Hence, we add row

$$x_6 - \frac{1}{7}x_3 - \frac{2}{7}x_5 = 1$$

to the tableau above:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	0	$z = \frac{30}{7}$
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	0	$\frac{13}{7}$
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	0	$\frac{9}{7}$
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	0	$\frac{31}{7}$
x_6	1	0	0	0	0	1	1

(The above tableau is a tableau for the second LP, after we add the cut (1) to our original LP relaxation. Call this new linear program (LP_2).)

Iteration 2:

Solve (LP_2): We can use dual simplex method to find the new optimal solution[1]. First, we need to do some row operations to make the last entry of the column of x_1 zero (subtract the first row from the last row):

	x_1	x_2	x_3	x_4	x_5	x_6	RHS	
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	0	$z = \frac{30}{7}$	
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	0	$\frac{13}{7}$	
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	0	$\frac{9}{7}$	
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	0	$\frac{31}{7}$	
x_6	0	0	$-\frac{1}{7}$	0	$-\frac{2}{7}$	1	$-\frac{6}{7}$	$\leftarrow$ dual simplex pivot

Then, 2 dual simplex pivots give the following optimal tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS	
	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	$z = \frac{7}{4}$	$\leftarrow$ source row for cut
x_1	1	0	0	0	0	1	1	
x_3	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	$\frac{5}{2}$	
x_2	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	$\frac{5}{4}$	
x_5	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{7}{4}$	

Add a Gomory cutting plane: Suppose we choose the objective row as the source row,

$$z - \frac{1}{4}x_4 - \frac{17}{4}x_6 = \frac{7}{4}.$$

So, the resulting cut is

$$z - x_4 - 5x_6 \leq 1.$$

Adding slack variable $x_7 \geq 0$ yields

$$z - x_4 - 5x_6 + x_7 = 1. \tag{2}$$

Hence, we add this row to the tableau above, producing:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
	1	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	0	$z = \frac{7}{4}$
x_1	0	1	0	0	0	0	1	0	1
x_3	0	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	0	$\frac{5}{2}$
x_2	0	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	0	$\frac{5}{4}$
x_5	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	0	$\frac{7}{4}$
x_7	1	0	0	0	-1	1	-5	1	1

We need to make the last entry of the column for z zero by subtracting the objective row from the last row, producing:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	
	1	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	0	$z = \frac{7}{4}$	
x_1	0	1	0	0	0	0	1	0	1	
x_3	0	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	0	$\frac{5}{2}$	
x_2	0	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	0	$\frac{5}{4}$	
x_5	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	0	$\frac{7}{4}$	
x_7	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	1	$-\frac{3}{4}$	$\leftarrow$ dual simplex pivot

[1] Alternatively, we can solve $(LP2)$ from scratch.

(The above tableau is a tableau for the third LP, after we add the cut (2) to (LP_2). Call this new linear program (LP_3).)

Iteration 3: As we did before, 1 dual simplex pivot solves the problem. We won't show the steps here, but it is very similar to what we did in Iteration 2. The optimal basic variables for (LP_3) (along with their values) are:

$$z = 1, \quad x_1 = 1, \quad x_3 = 4, \quad x_2 = 2, \quad x_5 = 1, \quad x_4 = 3.$$

This gives a feasible solution to the original integer program.

Observation. Observe that both cuts used can be expressed in terms of original variables x_1, x_2. We have

- For the <u>first cut</u>

$$\frac{6}{7} \le \frac{x_3}{7} + \frac{2x_5}{7} = \frac{3 - 3x_1 + 2x_2}{7} + \frac{2(5 - 2x_1 - x_2)}{7} = \frac{13}{7} - \frac{7x_1}{7} + \frac{0x_2}{7}.$$

  Hence, the first cut reads $x_1 \le 1$, or $x_1 + s_1 = 1$ with $s_1 \ge 0$.

- For the <u>second cut</u>

$$\frac{3}{4} \le \frac{x_4}{4} + \frac{s_1}{4} = \frac{-10 + 5x_1 + 4x_2}{4} + \frac{1 - x_1}{4} = -\frac{9}{4} + \frac{4x_1}{4} + \frac{4x_2}{4}.$$

  Hence, the second cut reads $x_1 + x_2 \ge 3$, or $x_1 + x_2 - s_2 = 3$ with $s_2 \ge 0$.

Thus, the solution process for the example above can be depicted graphically as follows: