Lecture 9: Feb 19, 2013

1 Dynamic Programming

1.1 Motivating example: the shortest-path problem

Consider the following input to the shortest-path problem.

This problem seems quite intimidating to solve. However, suppose that we were given the following information:

- The shortest path from s to 1 is known, with length ...
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Then, we can make the following observations.

Observation. Any path from s to t must involve one of the nodes ... right before arriving at t. In particular, a shortest path from s to t must involve a path from s to one of the nodes ..., then the edge from that node to t.

Observation. A shortest path from s to t must involve a shortest path from s to one of the nodes ..., then the edge from that node to t.

So, a shortest path from s to t can be found by considering the minimum of the following quantities:

$$\begin{aligned}
\text{The length of the shortest path from } s \text{ to followed by the edge}(,t) &= f^*(i) + c_{it}, \\
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\end{aligned}$$

Or the equivalent expression, the shortest path from s to t, denoted $f^*(t)$ is

$$f^*(t) = \min_{i \ s.t. \ (i,t) \in E} \{f^*(i) + c_{it}\},$$

where for each node i in the graph, $f^*(i)$ denote the shortest path from s to i.