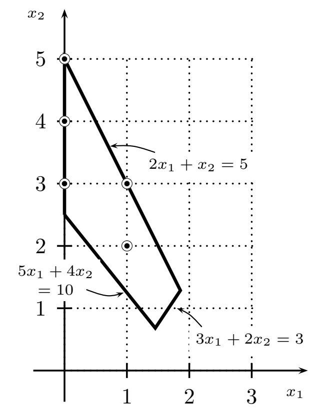
Lecture 19: April 2, 2013

1 Example

Consider the following IP problem

$$\begin{aligned}
\max\ z &= 3x_1 - x_2 \\
\text{s.t.}\quad & 3x_1 - 2x_2 \le 3 \\
& -5x_1 - 4x_2 \le -10 \\
& 2x_1 + x_2 \le 5 \\
& x_1,\ x_2 \ge 0, \quad \text{integer}
\end{aligned}$$



The feasible (integer) points, denoted $\odot$, along with the feasible region of the corresponding LP relaxation are depicted on the figure to the right.

Iteration 1:

Solve LP1:We solve the first LP relaxation. It turns out that the optimal LP solution tableau reads

	x_1	x_2	x_3	x_4	x_5	RHS
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	$z = \frac{30}{7}$
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	$\frac{13}{7}$
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	$\frac{9}{7}$
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	$\frac{31}{7}$

$\leftarrow$ source row for cut

Add a Gomory cutting plane: The first row is the source row (the row that we'll use to get the cut):

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_5 = \frac{13}{7},$$

so, the cut is

$$x_1 + 0x_3 + 0x_5 = 1,$$

Adding slack variable $x_6 \ge 0$ yields

$$x_1 + x_6 = 1. \tag{1}$$

Hence, we add row

$$x_6 - \frac{1}{7}x_3 - \frac{2}{7}x_5 = 1$$

to the tableau above:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	0	$z = \frac{30}{7}$
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	0	$\frac{13}{7}$
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	0	$\frac{9}{7}$
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	0	$\frac{31}{7}$
x_6	1	0	0	0	0	1	1

(The above tableau is a tableau for the second LP, after we add the cut (1) to our original LP relaxation. Call this new linear program (LP_2).)

Iteration 2:

Solve (LP_2): We can use dual simplex method to find the new optimal solution[1]. First, we need to do some row operations to make the last entry of the column of x_1 zero (subtract the first row from the last row):

	x_1	x_2	x_3	x_4	x_5	x_6	RHS	
	0	0	$-\frac{5}{7}$	0	$-\frac{3}{7}$	0	$z = \frac{30}{7}$	
x_1	1	0	$\frac{1}{7}$	0	$\frac{2}{7}$	0	$\frac{13}{7}$	
x_2	0	1	$-\frac{2}{7}$	0	$\frac{3}{7}$	0	$\frac{9}{7}$	
x_4	0	0	$-\frac{3}{7}$	1	$\frac{22}{7}$	0	$\frac{31}{7}$	
x_6	0	0	$-\frac{1}{7}$	0	$-\frac{2}{7}$	1	$-\frac{6}{7}$	$\leftarrow$ dual simplex pivot

Then, 2 dual simplex pivots give the following optimal tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS	
	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	$z = \frac{7}{4}$	$\leftarrow$ source row for cut
x_1	1	0	0	0	0	1	1	
x_3	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	$\frac{5}{2}$	
x_2	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	$\frac{5}{4}$	
x_5	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{7}{4}$	

Add a Gomory cutting plane: Suppose we choose the objective row as the source row,

$$z - \frac{1}{4}x_4 - \frac{17}{4}x_6 = \frac{7}{4}.$$

So, the resulting cut is

$$z - x_4 - 5x_6 \leq 1.$$

Adding slack variable $x_7 \geq 0$ yields

$$z - x_4 - 5x_6 + x_7 = 1. \tag{2}$$

Hence, we add this row to the tableau above, producing:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
	1	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	0	$z = \frac{7}{4}$
x_1	0	1	0	0	0	0	1	0	1
x_3	0	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	0	$\frac{5}{2}$
x_2	0	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	0	$\frac{5}{4}$
x_5	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	0	$\frac{7}{4}$
x_7	1	0	0	0	-1	1	-5	1	1

We need to make the last entry of the column for z zero by subtracting the objective row from the last row, producing:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	
	1	0	0	0	$-\frac{1}{4}$	0	$-\frac{17}{4}$	0	$z = \frac{7}{4}$	
x_1	0	1	0	0	0	0	1	0	1	
x_3	0	0	0	1	$-\frac{1}{2}$	0	$\frac{11}{2}$	0	$\frac{5}{2}$	
x_2	0	0	1	0	$-\frac{1}{4}$	0	$\frac{5}{4}$	0	$\frac{5}{4}$	
x_5	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	0	$\frac{7}{4}$	
x_7	0	0	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	1	$-\frac{3}{4}$	$\leftarrow$ dual simplex pivot

[1] Alternatively, we can solve ($LP2$) from scratch.

(The above tableau is a tableau for the third LP, after we add the cut (2) to (LP_2). Call this new linear program (LP_3).)

Iteration 3: As we did before, 1 dual simplex pivot solves the problem. We won't show the steps here, but it is very similar to what we did in Iteration 2. The optimal basic variables for (LP_3) (along with their values) are:

$$z = 1, \quad x_1 = 1, \quad x_3 = 4, \quad x_2 = 2, \quad x_5 = 1, \quad x_4 = 3.$$

This gives a feasible solution to the original integer program.

Observation. Observe that both cuts used can be expressed in terms of original variables x_1, x_2. We have

- For the <u>first cut</u>

$$\frac{6}{7} \leq \frac{x_3}{7} + \frac{2x_5}{7} = \frac{3 - 3x_1 + 2x_2}{7} + \frac{2(5 - 2x_1 - x_2)}{7} = \frac{13}{7} - \frac{7x_1}{7} + \frac{0x_2}{7}.$$

  Hence, the first cut reads $x_1 \leq 1$, or $x_1 + s_1 = 1$ with $s_1 \geq 0$.

- For the <u>second cut</u>

$$\frac{3}{4} \leq \frac{x_4}{4} + \frac{s_1}{4} = \frac{-10 + 5x_1 + 4x_2}{4} + \frac{1 - x_1}{4} = -\frac{9}{4} + \frac{4x_1}{4} + \frac{4x_2}{4}.$$

  Hence, the second cut reads $x_1 + x_2 \geq 3$, or $x_1 + x_2 - s_2 = 3$ with $s_2 \geq 0$.

Thus, the solution process for the example above can be depicted graphically as follows: