ORIE 3310 - Spring 2013
Lecture 1 (T 22 Jan)

- Write on board:

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  OH: T 10:30 - 12:30, Th 12:30 - 2:30

- Ask students to fill out bio sheets.
- Go over syllabus.

- In optimization 1, we learn about Linear Programming and an algorithm for solving them, the simplex method.
- Let us briefly recall what an LP problem looks like.

on slides

Ex: CTB sells bagels and cupcakes.

Ingredients:

	1 dz Bagel	1 dz Cupcake	Daily Amount Available
Eggs	3	6	50
Flour	7	5	100
Butter	3	4	75

CTB has some regulars who come for bagels, so must have at least 3 dozens bagels.

Bagel's profits: \$6 per dozen,
cupcake's profits: \$8 per dozen.

How many dozens of bagels and cupcakes should CTB produce each day to maximize total profit?

Formulate as LP:

1. Decision variables: X_1 = number of dozens of bagels
   X_2 = number of dozens of cupcakes
2. Objective function: A linear function in terms of dec. vars which we minimize or maximize:
   Total profit $= 6X_1 + 8X_2$.

③ Constraints: Linear equalities or inequalities in terms of the dec. va.

$$3x_1 + 6x_2 \le 50$$
$$7x_1 + 5x_2 \le 100$$
$$3x_1 + 4x_2 \le 75$$
$$x_1 \ge 3$$
$$x_1, x_2 \ge 0.$$

So, the LP:

$$\begin{aligned}
\text{Max} \quad & 6x_1 + 8x_2 \\
\text{s.t.} \quad & 3x_1 + 6x_2 \le 50 \\
& 7x_1 + 5x_2 \le 100 \\
& 3x_1 + 4x_2 \le 75 \\
& x_1 \ge 3 \\
& x_1, x_2 \ge 0
\end{aligned}$$

Can add slack variables so that the constraints are equality constraints:

$$\begin{aligned}
\text{Max} \quad & 6x_1 + 8x_2. \\
\text{s.t.} \quad & 3x_1 + 6x_2 + S_1 = 50 \\
& 7x_1 + 5x_2 + S_2 = 100 \\
& 3x_1 + 4x_2 + S_3 = 75 \\
& x_1 - S_4 = 3 \\
& x_1, x_2, S_1, S_2, S_3, S_4 \ge 0.
\end{aligned}$$

So in general, LP's look like:

$$\begin{aligned}
\text{(Min or) Max} \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \ge 0.
\end{aligned}$$

In this example, $c = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 6 & 1 & 0 & 0 & 0 \\ 7 & 5 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 50 \\ 100 \\ 75 \\ 3 \end{pmatrix}$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_6 \end{pmatrix}.$

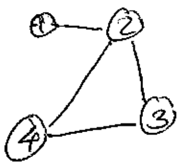
- The vectors and matrix: c, A, b are <u>inputs</u> to our linear program.
- The <u>simplex method</u> is an algorithm which allows us to find an optimal solution to a linear program.

- While the focus of Opt 1 is on linear programming, what we'll do this semester in Opt 2 is ~~a sur~~
  - A survey of a selection of other areas and methods ~~of~~/in optimization.
  - The 4 topics are chosen because of their importance both in practice / industry / application as well as in the theory of opt'n.
  - Moreover! Although we can think of them as 4 "different" topics, it is actually more useful (for learning) and is just cooler to recognize various connections among them, and between them and LP ideas.
- Our first topic, Network flows, is a great starting point:
  Many problems can be modeled as a problem on a network, even when they don't seem to involve any graphs / networks.
  Also, many network problems can be formulated as an LP.

- What do we mean by <u>a network</u>?

  A <u>network</u>, or a <u>graph</u> is a collection of <u>nodes</u> ("vertices") and <u>edges</u> ("arcs") that connect pairs of nodes.

  E.g.



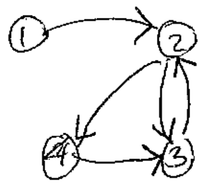
  $N = \{1, 2, 3, 4\}$ set of nodes.

  $E = \{\{1,2\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ set of edges.

  Graph $G = (N, E)$.

  The example above is an <u>undirected</u> graph because the edges do not have a direction.

The following is a directed graph:



$G=(N,E)$

$E=\{(1,2),(2,3),(3,2),(2,4),(4,3)\}$.
$N=\{1,2,3,4\}$.

Note: the edge $(2,3)$ is different from $(3,2)$.

The first network optimization problem that we'll look at will be a problem on a directed graph (i.e. it takes a dir. graph as an input).

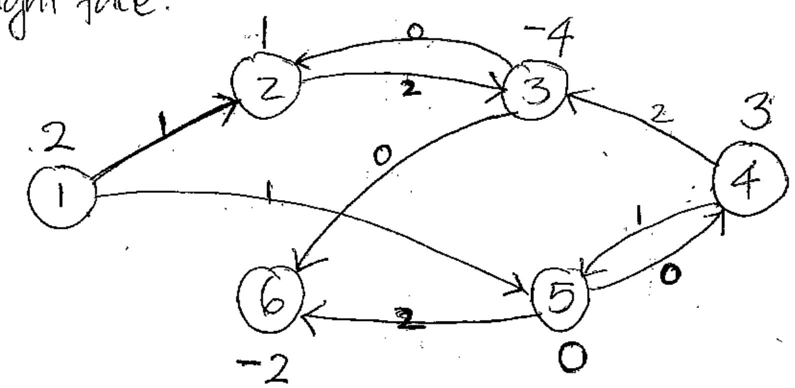
The Minimum-cost network flow problem.

- I will first describe the problem "informally"
  - Consider a problem that a company such as Amazon.com might face:

Ex.

6 nodes
9 edges.

$\boxed{\text{Total cost} = \$8}$



On each edge:
cost per unit = \$1
Capacity = 2 units

- The company owns several warehouses that have a supply of the AMPL books.
- They also have data on customers who ordered the books from various college towns.
- The directed graph above represents this data, as well as the available shipping routes / connections

Nodes = locations
Edges = available shipping connections.

We specify a "supply value" on each node.
(positive = supply node,
negative = demand node,
zero = transit node.)

Also have data:
cost per unit of shipment along each edge
capacity on each edge.

Problem: Minimizing total shipping cost
while satisfying all demands, and the capacity constraints

Example solution: see graph (highlighted)

Note: In our example, total supply = $3+2+1=6$.
total demand $= +2 + +4 = 6$.
so, we ~~can~~ might be able to satisfy all demand

∴ For the problem to be solvable, it is necessary that
total supply $\geq$ total demand.

But for our min-cost flow model, we'll make a stronger assumption on our input data:

Assumption: total supply = total demand.
$\therefore \sum$ supply values $= 0$.

So, here is the minimum-cost network flow problem ("model") formally:

- Given the following inputs:
  1. A directed graph $G = (N, E)$
  2. Supply values $b_i \ \forall i \in N$
  3. Edge costs (per unit) $c_{ij} \ \forall (i,j) \in E$
  4. Edge capacity $u_{ij} \ \forall (i,j) \in E$ ≥ 0.
- Find an optimal "flow" (i.e. # units to be shipped on each edge) that minimizes total cost while satisfying capacity and supply constraints:

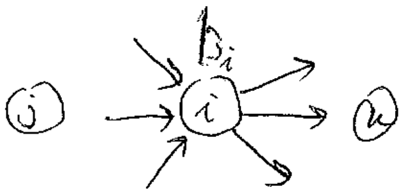
① Capacity constraints: on each edge $(i,j) \in E$:

\# units shipped along edge (i,j) $\leq u_{ij}$

② Supply constraints: on each node $i \in N$:

net flow out of node i = supply value at node i

$\therefore$ flow out of i − flow into i = b_i.



Assumption: $\sum b_i = 0$.

This looks like a problem whose objective and constraints are linear in terms of the flow values.

Formulate as an LP:

① Dec. variables: for each $(i,j) \in E$: x_{ij} = # units shipped along edge (i,j)
= "flow value on edge (i,j)".

E.g. $x = \begin{pmatrix} x_{12} \\ x_{23} \\ x_{32} \\ \vdots \end{pmatrix}$ ← a vector

② Objective: Minimize total cost

$$\text{Total cost} = \sum_{(i,j) \in E} c_{ij} x_{ij}$$

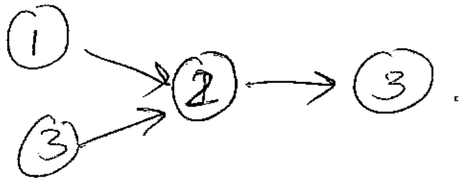
E.g.: $1 \cdot x_{12} + 1 \cdot x_{23} + \ldots$ = total cost.

③ Constraints

① Cap. constraints: $x_{ij} \leq u_{ij} \quad \forall (i,j) \in E$.

② Flow conservation constraints.

Eg: node 2:



$$\underbrace{X_{23}}_{\text{flow out of 2}} - (\underbrace{X_{12} + X_{32}}_{\text{flow into 2}}) \quad \text{must equal } S_2 = 1.$$

$$\therefore \quad X_{23} - (X_{12} + X_{32}) = 1.$$

(A couple more example: demand & transit nodes.)

$\therefore$ In general:

$$\sum_{(i,j)\in E} X_{ij} - \sum_{(k,i)\in E} X_{ki} = b_i \quad \forall i \in N.$$

∘ LP formulation of min-cost network flow:

$$\boxed{\begin{aligned} &\text{Min} \sum_{(i,j)\in E} C_{ij} X_{ij}. \\ &\text{s.t.} \quad X_{ij} \le u_{ij} \qquad \forall (i,j)\in E. \\ &\qquad \sum_{(i,j)\in E} X_{ij} - \sum_{(k,i)\in E} X_{ki} = b_i \qquad \forall i \in E \\ &\qquad X_{ij} \ge 0. \end{aligned}}$$

→ AMPL Demo.

Next Class: Integrality thm for min-cost flow.