Correctional Note to
Existence and Uniqueness of Semimartingale Reflecting Brownian Motions in Convex Polyhedrons,
by J. G. Dai and R. J. Williams,
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Lemma 4.1 of this paper need not always hold as stated (e.g., for $S = \{x \in \mathbf{R} : 0 \le x \le 1\}$ and $\epsilon = 2$, the left member of (4.2) is S and the right member of (4.2) is empty). However, (4.2) does hold for all ε sufficiently small. The resultant corrected form of Lemma 4.1 is given below. As explained further below, this lemma implies a local oscillation inequality — a corrected version of Lemma 4.3 — which is also stated below. The corrected forms of Lemmas 4.1 and 4.3 suffice for the proofs of the main results in the paper. Consequently those results on existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedrons remain valid.

Correction to Lemma 4.1.

The corrected form of Lemma 4.1 is as follows.

Lemma 4.1. *There is $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$,*

$$S = \bigcup_{\mathbf{K} \in \mathcal{C}} F^{\varepsilon}_{\mathbf{K}},$$

where $\mathcal{C}$ denotes the collection of subsets of $\mathbf{J}$ consisting of all maximal sets in $\mathbf{J}$ together with the empty set.

Correction to the Proof. The original proof begins with the correct statement that for any $\varepsilon \ge 0$,

$$S = \bigcup_{\mathbf{L} \subset \mathbf{J}} G^{\varepsilon}_{\mathbf{L}},$$

where $\mathbf{L}$ ranges over all subsets of $\mathbf{J}$, including the empty set, and

$$G^{\varepsilon}_{\mathbf{L}} \equiv \{x \in \mathbf{R}^d : 0 \le n_i \cdot x - b_i \le \varepsilon \text{ for all } i \in \mathbf{L},\ n_i \cdot x - b_i > \varepsilon \text{ for all } i \in \mathbf{J} \setminus \mathbf{L}\}.$$

The error in the proof occurs in the next line which makes the (false) assertion that if $\emptyset \ne \mathbf{L} \subset \mathbf{J}$ is not maximal, then there is a maximal $\mathbf{K} \supset \mathbf{L}$ such that $F_{\mathbf{L}} = F_{\mathbf{K}}$. This assertion is true (by the definition of a maximal set) *provided* that $F_{\mathbf{L}} \ne \emptyset$. Now, we only need the conclusion of the assertion for $\mathbf{L} \ne \emptyset$ such that $G^{\varepsilon}_{\mathbf{L}} \ne \emptyset$. Consequently, the rest of the proof is valid provided that $F_{\mathbf{L}} \ne \emptyset$ whenever $G^{\varepsilon}_{\mathbf{L}} \ne \emptyset$ and $\mathbf{L} \ne \emptyset$. This is true for all ε sufficiently small, as we now show.

Let

$$\varepsilon_1 = \inf\{d(F_{\mathbf{M}}, F_i) : \emptyset \ne \mathbf{M} \subset \mathbf{J},\ F_{\mathbf{M}} \ne \emptyset,\ i \in \mathbf{J} \setminus \mathbf{M},\ F_{\mathbf{M}} \cap F_i = \emptyset\},$$

where $d(F_{\mathbf{M}}, F_i)$ denotes the distance between $F_{\mathbf{M}}$ and F_i, and the infimum of the empty set is $+\infty$. It follows from Lemma 1 below that $\varepsilon_1 > 0$, although it may take the value $+\infty$. Let $\varepsilon_0 \in (0, \infty)$ such that

$$\varepsilon_0 < \frac{\varepsilon_1}{Cm},$$

where C is the constant appearing in Lemma B.1 and $m = |\mathbf{J}|$. Suppose that $0 \le \varepsilon < \varepsilon_0$ and let $\emptyset \ne \mathbf{L} \subset \mathbf{J}$ such that $G^{\varepsilon}_{\mathbf{L}} \ne \emptyset$. For a proof by contradiction, suppose that $F_{\mathbf{L}} = \emptyset$. Then, since $\mathbf{L} \ne \emptyset$ and $F_j \ne \emptyset$ for all $j \in \mathbf{J}$, there is $\mathbf{M} \subset \mathbf{L}$ such that $\mathbf{M} \ne \emptyset$, $\mathbf{M} \ne \mathbf{L}$, $F_{\mathbf{M}} \ne \emptyset$ and $F_{\mathbf{M}} \cap F_i = \emptyset$

for some $i \in \mathbf{L} \setminus \mathbf{M}$. Let $x^* \in G^{\varepsilon}_{\mathbf{L}}$. Then, $x^* \in S$ and by Lemma B.1, since $n_i \cdot x^* - b_i \leq \varepsilon$ for all $i \in \mathbf{L}$, we have

$$d(x^*, F_{\mathbf{M}}) \leq C|\mathbf{M}|\varepsilon,$$
$$d(x^*, F_i) \leq C\varepsilon,$$

and so

$$d(F_{\mathbf{M}}, F_i) \leq d(x^*, F_{\mathbf{M}}) + d(x^*, F_i) \leq C|\mathbf{L}|\varepsilon \leq Cm\varepsilon < \varepsilon_1.$$

However, this contradicts the definition of ε_1. This completes the proof of the desired assertion and the corrected version of Lemma 4.1 as stated above follows. □

Lemma 1. *Suppose that* $\emptyset \neq \mathbf{M} \subset \mathbf{J}$ *such that* $F_{\mathbf{M}} = \cap_{j \in \mathbf{M}} F_j \neq \emptyset$ *and* $i \in \mathbf{J} \setminus \mathbf{M}$ *such that* $F_{\mathbf{M}} \cap F_i = \emptyset$. *Then,*

$$d(F_{\mathbf{M}}, F_i) > 0.$$

Proof. We note first that $F_{\mathbf{M}} \cap H_i = \emptyset$, where $H_i = \{x \in \mathbf{R}^d : n_i \cdot x = b_i\}$. If this were not so, then there would be $x \in F_{\mathbf{M}} \cap H_i$. This x would be in S, since $F_{\mathbf{M}} \subset S$. Then, since $F_i = S \cap H_i$, x would be in F_i, and since $x \in F_{\mathbf{M}}$ also, this would contradict $F_{\mathbf{M}} \cap F_i = \emptyset$.

Since $F_i \subset H_i$, $d(F_{\mathbf{M}}, F_i) \geq d(F_{\mathbf{M}}, H_i)$, and so it suffices to prove that $d(F_{\mathbf{M}}, H_i) > 0$. For this, note that since $F_{\mathbf{M}}$ is convex and does not meet H_i, $F_{\mathbf{M}}$ must lie on one side of the hyperplane H_i. Without loss of generality, we assume that $n_i \cdot x > b_i$ for all $x \in F_{\mathbf{M}}$. Then, the distance from $F_{\mathbf{M}}$ to the hyperplane H_i is given by the "perpendicular" distance:

$$\inf\{n_i \cdot x - b_i : x \in F_{\mathbf{M}}\}.$$

Since $F_{\mathbf{M}}$ is not empty, either the optimal cost of this linear programming problem is $-\infty$ or there is an $x^* \in F_{\mathbf{M}}$ that achieves the infimum (cf. D. Bertsimas and J. N. Tsitsiklis, Linear Optimization, Athena Scientific Press, Belmont, MA, 1997; Corollary 2.3, page 67). Since $n_i \cdot x > b_i$ for all $x \in F_{\mathbf{M}}$, it follows that the former option cannot occur and an optimal $x^* \in F_{\mathbf{M}}$ exists with

$$d(F_{\mathbf{M}}, H_i) = n_i \cdot x^* - b_i > 0,$$

as desired. □

Implications for other results.

Lemma 4.1 was used to prove Lemma 4.3. In view of the correction described above, only a local version of Lemma 4.3, as stated below, is true. This can be proved in a similar manner to that in the original text, provided one restricts the size ε of the oscillations of x to be sufficiently small. In particular, for parts (b) and (c) of the proof, one needs that $C_1\varepsilon < \varepsilon_0$ where ε_0 is the value indicated in the corrected statement of Lemma 4.1 above. (We note here that there is a non-critical typo in the formula for C_2 in part (c), it should read $C_2 = C_1 C^2 m^2$.)

Lemma 4.3. *There exist constants* $\kappa > 0$ *and* $\delta > 0$ *that depend only on* (S, R) *such that for any* $T > 0$, $x \in C([0,T], \mathbf{R}^d)$ *with* $x(0) \in S$, *and an* (S,R)*-regulation* (y, z) *of* x *over* $[0,T]$, *the following holds for each interval* $[t_1, t_2] \subset [0,T]$*:*

$$\mathrm{Osc}(y, [t_1, t_2]) \leq \kappa\, \mathrm{Osc}(x, [t_1, t_2]) \quad \textit{and} \quad \mathrm{Osc}(z, [t_1, t_2]) \leq \kappa\, \mathrm{Osc}(x, [t_1, t_2]),$$

whenever $\mathrm{Osc}(x, [t_1, t_2]) \leq \delta$.

The main results of the paper remain valid, for when Lemmas 4.1 and 4.3 are used in deriving those results, only local versions of them are needed. In particular, these lemmas are used in establishing tightness and for that purpose the local, corrected versions cited here suffice.

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