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EXISTENCE AND UNIQUENESS OF SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS IN CONVEX POLYHEDRONS*

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Abstract. We consider the problem of existence and uniqueness of semimartingale reflecting Brownian motions (SRBM's) in convex polyhedrons. Loosely speaking, such a process has a semimartingale decomposition such that in the interior of the polyhedron the process behaves like a Brownian motion with a constant drift and covariance matrix, and at each of the $(d-1)$-dimensional faces that form the boundary of the polyhedron, the bounded variation part of the process increases in a given direction (constant for any particular face), so as to confine the process to the polyhedron. For historical reasons, this "pushing" at the boundary is called instantaneous reflection. For simple convex polyhedrons, we give a necessary and sufficient condition on the geometric data for the existence and uniqueness of an SRBM. For nonsimple convex polyhedrons, our condition is shown to be sufficient. It is an open question as to whether our condition is also necessary in the nonsimple case. From the uniqueness, it follows that an SRBM defines a strong Markov process. Our results are applicable to the study of diffusions arising as heavy traffic limits of multiclass queueing networks and in particular, the nonsimple case is applicable to multiclass fork and join networks. Our proof of weak existence uses a patchwork martingale problem introduced by T. G. Kurtz, whereas uniqueness hinges on an ergodic argument similar to that used by L. M. Taylor and R. J. Williams to prove uniqueness for SRBM's in an orthant.

Key words. semimartingale reflecting Brownian motion, diffusion process, nonsimple convex polyhedron, completely-$\mathcal{S}$ matrix, martingale problems, multiclass queueing networks, fork and join networks

1. Introduction. This paper is concerned with the existence and uniqueness of a class of semimartingale reflecting Brownian motions which live in a d-dimensional convex polyhedron S $(d \geqq 1)$. The polyhedron is defined in terms of m $(m \geqq 1)$ d-dimensional unit vectors $\{n_i,\ i \in \mathbf{J}\}$, $\mathbf{J} \equiv \{1, \ldots, m\}$, and an m-dimensional vector $b = (b_1, \ldots, b_m)'$, where prime denotes transpose. (Hereafter vectors are taken to be column vectors.) The state space S is defined by

$$S \equiv \{x \in \mathbf{R}^d\colon\ n_i \cdot x \geqq b_i \ \text{ for all } \ i \in \mathbf{J}\}, \tag{1.1}$$

where $n_i \cdot x = n_i' x$ denotes the inner product of the vectors n_i and x. It is assumed that the interior of S is nonempty and that the set $\{(n_1, b_1), \ldots, (n_m, b_m)\}$ is minimal in the sense that no proper subset defines S. That is, for any strict subset $\mathbf{K} \subset \mathbf{J}$, the set $\{x \in \mathbf{R}^d\colon\ n_i \cdot x \geqq b_i\ \forall\ i \in \mathbf{K}\}$ is strictly larger than S. This is equivalent to the assumption that each of the faces

$$F_i \equiv \{x \in S\colon\ n_i \cdot x = b_i\}, \qquad i \in \mathbf{J}, \tag{1.2}$$

has dimension $d-1$ (cf. [7, Thm. 8.2]). As a consequence, n_i is the unit normal to F_i that points into the interior of S.

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Given a d-dimensional vector θ, a $d \times d$ symmetric, positive definite matrix Γ, and a $d \times m$ matrix R, we shall define a semimartingale reflecting Brownian motion associated with the data (S, θ, Γ, R). For this note that a triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ will be called a *filtered space* if Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω, and $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t,\ t \geqq 0\}$ is an increasing family of sub-σ-fields of $\mathcal{F}$, i.e., a filtration. If, in addition, P is a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ will be called a filtered probability space.

DEFINITION 1.1. For $x \in S$, a *semimartingale reflecting Brownian motion* (abbreviated as SRBM) associated with the data (S, θ, Γ, R) that starts from x is an $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ such that

$$Z = X + RY, \tag{1.3}$$

where

(i) Z has continuous paths in S, P_x-a.s.,

(ii) under P_x, X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathcal{F}_t,\ t \geqq 0\}$ is a martingale and $X(0) = x$, P_x-a.s.,

(iii) Y is an $\{\mathcal{F}_t\}$-adapted, m-dimensional process such that P_x-a.s. for each $i \in \{1, \dots, m\}$, the ith component Y_i of Y satisfies

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and nondecreasing,

(c) Y_i can increase only when Z is on the face F_i, i.e., $\int_0^t 1_{F_i}(Z(s))\, dY_i(s) = Y_i(t)$ for all $t \geqq 0$.

An SRBM associated with the data (S, θ, Γ, R) is an $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z together with a family of probability measures $\{P_x, x \in S\}$ defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that (1.3) holds and for each $x \in S$, (i)–(iii) of Definition 1.1 hold.

Remarks. 1. Note that to allow flexibility in our definition of an SRBM that starts from x, we have not assumed that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\},\ P_x)$ is complete and we have only required (Z, Y) to be continuous P_x-a.s. However, one can always replace $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ with its completion, obtained by completing the probability space $(\Omega, \mathcal{F}, P_x)$ and augmenting the filtration $\{\mathcal{F}_t\}$ with the resulting P_x-null sets. Then by modifying Z and Y on a P_x-null set, one can make these processes continuous everywhere and such that the properties described in Definition 1.1 still hold. One can also replace $\mathcal{F}_t$ with $\mathcal{F}_{t+} \equiv \bigwedge_{s>t} \mathcal{F}_s$ to make the filtration right continuous. Indeed, in what follows, when we wish to apply the theory of stochastic calculus or other general theory requiring the "usual conditions" on a fixed probability space, we shall assume that the probability space has been completed and the filtration augmented and then often made right continuous in the manner mentioned above. Since the final results obtained in this setting can be translated back to the original situation, this does not affect their validity.

2. Another way of ensuring that all paths of (Z, Y) are continuous is to consider the canonical processes and natural σ-field and filtration on the (Z, Y) path space with the probability measure induced there by (Z, Y) under P_x. One advantage of this approach is that the same processes can be used for different starting points and all of the probability measures are defined on the same measurable space. In this way, one can obtain an SRBM as described at the end of Definition 1.1 (see Theorem 1.3).

DEFINITION 1.2. For each $\varnothing \neq \mathbf{K} \subset \mathbf{J}$, define $F_{\mathbf{K}} = \cap_{i \in \mathbf{K}} F_i$. Let $F_{\varnothing} = S$. A set $\mathbf{K} \subset \mathbf{J}$ is *maximal* if $\mathbf{K} \neq \varnothing$, $F_{\mathbf{K}} \neq \varnothing$, and $F_{\mathbf{K}} \neq F_{\widetilde{\mathbf{K}}}$ for any $\widetilde{\mathbf{K}} \supset \mathbf{K}$ such that $\widetilde{\mathbf{K}} \neq \mathbf{K}$.

In this paper, we make the following assumption on R and S. Here v_i denotes the ith column of the matrix R, for each $i \in \mathbf{J}$.

ASSUMPTION 1.1. For each maximal $\mathbf{K} \subset \mathbf{J}$,

(S.a) there is a positive linear combination $v = \sum_{i \in \mathbf{K}} a_i v_i \quad (a_i > 0 \; \forall i \in \mathbf{K})$ of the $\{v_i,\ i \in \mathbf{K}\}$ such that $n_i \cdot v > 0$ for all $i \in \mathbf{K}$;

(S.b) there is a positive linear combination $\eta = \sum_{i \in \mathbf{K}} c_i n_i \quad (c_i > 0 \; \forall i \in \mathbf{K})$ of the $\{n_i,\ i \in \mathbf{K}\}$ such that $\eta \cdot v_i > 0$ for all $i \in \mathbf{K}$.

The labels **(S.a)** and **(S.b)** stand for $\mathcal{S}$-condition (a) and (b), respectively. The origin of these labels becomes apparent when the conditions are written in matrix form as below. For a vector x, the notation $x > 0$ will mean that all coordinates of x are strictly positive, and the notation $x \geqq 0$ will mean that all coordinates of x are non-negative.

DEFINITION 1.3. A matrix A is called an $\mathcal{S}$ matrix if there is a vector $x \geqq 0$ such that $Ax > 0$. (See [18] for more details on $\mathcal{S}$ matrices, which are named after Stiemke.)

Remark. It is easy to see by perturbation that in the definition of an $\mathcal{S}$ matrix, x may be chosen such that $x > 0$.

Let N denote the $m \times d$ matrix whose ith *row* is given by the row vector n_i' for each $i \in \mathbf{J}$. For an $m \times m$ matrix A and $\mathbf{K} \subset \mathbf{J}$, let $A_{\mathbf{K}}$ denote the $|\mathbf{K}| \times |\mathbf{K}|$ matrix obtained from A by deleting those rows and columns with indices in $\mathbf{J} \backslash \mathbf{K}$.

Conditions **(S.a)** and **(S.b)** are equivalent to the following:

(S.a) The matrix $(NR)_{\mathbf{K}}$ is an $\mathcal{S}$ matrix;

(S.b) The matrix $(NR)_{\mathbf{K}}'$ is an $\mathcal{S}$ matrix.

DEFINITION 1.4. The convex polyhedron S is *simple* if for each $\mathbf{K} \subset \mathbf{J}$ such that $\mathbf{K} \neq \varnothing$ and $F_{\mathbf{K}} \neq \varnothing$, exactly $|\mathbf{K}|$ distinct faces contain $F_{\mathbf{K}}$.

This definition is a direct generalization of the one given at the top of p. 80 of [7] for a *bounded* simple polyhedron, i.e., a simple polytope. Using the definition of a maximal set, one can check that S is simple if and only if either of the following holds:

(a) every nonempty subset of a maximal set is maximal,

(b) $\mathbf{K} \subset \mathbf{J}$ is maximal whenever $\mathbf{K} \neq \varnothing$ and $F_{\mathbf{K}} \neq \varnothing$.

A point $x_0 \in S$ is a *vertex* of S if $F_{\mathbf{K}} = \{x_0\}$ for some $\mathbf{K} \subset \mathbf{J}$. If S is simple, precisely d faces meet at any vertex of S.

PROPOSITION 1.1. *Suppose that S is simple. Then* **(S.a)** *holds for all maximal* $\mathbf{K} \subset \mathbf{J}$ *if and only if* **(S.b)** *holds for all maximal* $\mathbf{K} \subset \mathbf{J}$.

Proof. Suppose that **(S.a)** holds for all maximal $\mathbf{K} \subset \mathbf{J}$. Combined with the simple property of S this implies that for any maximal $\mathbf{L} \subset \mathbf{J}$, $(NR)_{\mathbf{L}}$ is completely-$\mathcal{S}$, i.e., each principal submatrix of $(NR)_{\mathbf{L}}$ is an $\mathcal{S}$ matrix. It was shown in Lemma 3 of [34] that a square matrix A is completely-$\mathcal{S}$ if and only if A' is completely-$\mathcal{S}$. It follows that $(NR)_{\mathbf{L}}'$ is completely-$\mathcal{S}$ for all maximal $\mathbf{L} \subset \mathbf{J}$, and hence **(S.b)** holds for all maximal $\mathbf{K} \subset \mathbf{J}$. A symmetric argument proves the converse part of the proposition.

For the case $S = \mathbf{R}_+^d$, it was shown in Theorem 2 of [34] that **(S.a)** holding for all maximal $\mathbf{K} \subset \mathbf{J}$ is *necessary* for the existence of an SRBM starting from each point $x \in \mathbf{R}_+^d$. In fact, the argument in Theorem 2 of [34] can be readily adapted to the more general state spaces considered here to prove the following.

PROPOSITION 1.2. *Suppose that for each $x \in S$ there exists an SRBM associated with (S, θ, Γ, R) that starts from x. Then* **(S.a)** *holds for all maximal* $\mathbf{K} \subset \mathbf{J}$.

Combining this with Proposition 1.1 we see that in the case where S is simple, Assumption 1.1 is necessary for the existence of an SRBM starting from each point in S. It is still an open question as to whether **(S.b)** holding for all maximal $\mathbf{K} \subset \mathbf{J}$ is necessary for such existence in the case when S is not simple.

Henceforth, when we refer to conditions **(S.a)** and **(S.b)**, we shall mean that they hold for all maximal $\mathbf{K} \subset \mathbf{J}$.

As a complement to the above discussion about necessity, we will show that Assumption 1.1 is sufficient for the existence and uniqueness in law of an SRBM starting from each point in S. This is the main result of this paper. For the precise statement of this result, let $\mathbf{C} = C([0,\infty),\ S \times \mathbf{R}_+^m) = \{(z,y)\colon\ z,y$ are continuous functions from $[0,\infty)$ into S, $\mathbf{R}_+^m$, respectively$\}$, $\mathcal{M} = \sigma\{(z,y)(s)\colon\ 0 \leqq s < \infty,\ (z,y) \in \mathbf{C}\}$, and for each $t \geqq 0$, $\mathcal{M}_t = \sigma\{(z,y)(s)\colon\ 0 \leqq s \leqq t,\ (z,y) \in \mathbf{C}\}$.

THEOREM 1.3. *Suppose that Assumption* 1.1 *holds. Fix $x \in S$. There exists an SRBM associated with (S,θ,Γ,R) that starts from x. Let Z defined on some filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P_x)$ be such an SRBM and let Y denote its associated "pushing process" with properties as described in Definition* 1.1. *Let Q_x denote the probability measure induced on $(\mathbf{C},\mathcal{M})$ by (Z,Y) under P_x:*

$$Q_x(A) = P_x\big((Z,Y) \in A\big) \qquad \textit{for all} \quad A \in \mathcal{M}. \tag{1.4}$$

Then, Q_x is unique and hence the law of any SRBM, together with its associated pushing process, for the data $(S,\ \theta,\Gamma,R)$ and starting point x is unique.

The canonical process $z(\cdot)$ together with the family of probability measures $\{Q_x,\ x \in S\}$ defines an SRBM on $(\mathbf{C},\mathcal{M},\{\mathcal{M}_t\})$, where for the decomposition (1.3) *one can take $Y(\cdot) = y(\cdot)$ and $X(\cdot) = z(\cdot) - Ry(\cdot)$. The family $\{Q_x,\ x \in S\}$ is Feller continuous and together with the canonical process $z(\cdot)$ defines a strong Markov process.*

When $S = \mathbf{R}_+^d$, the sufficiency of **(S.a)** for existence and uniqueness in law of an SRBM was shown in [37]. By using linear transformations and a patching and localization procedure, the results of [37] could be used to prove Theorem 1.3 for SRBM's in simple convex polyhedrons. Our aim in this paper is to prove the general existence and uniqueness result Theorem 1.3, which applies even in the case of nonsimple convex polyhedrons. Our proof of existence is different from that in [37]. We use the device of patchwork and constrained martingale problems introduced by Kurtz [28]. The construction of a suitable test function, contained in Appendix B, is crucial for this. In addition to allowing us to treat nonsimple polyhedrons, this approach is global and so it allows us to avoid the need to patch together measures on path space to obtain existence. Our proof of uniqueness employs localization to the case of a cone and then uses the same basic idea as in [37]. However, a variety of details need to be verified for the more general situation considered here. In particular, generalizations of the oscillation estimate of Bernard and El Kharroubi [4] and the boundary property of Reiman and Williams [34] are established in § 4.

One of the primary motivations for studying SRBM's is that they have been proposed as approximate models for queueing networks under conditions of heavy traffic. SRBM's in the d-dimensional orthant $\mathbf{R}_+^d$ have been proposed as approximations to open networks of d stations without buffer constraints (see [32], [31], [23], [20], and [21], [22]). For closed d-station networks, the proposed approximating SRBM's live in the d-dimensional simplex $\{x \in \mathbf{R}_+^d\colon\ x_1 + \cdots + x_d = 1\}$ (see [24], [9], [12]). Such an SRBM, though not defined in this paper, can be shown to be equivalent to one considered here, by projecting onto the $(d-1)$-dimensional solid simplex $\{x \in \mathbf{R}_+^{d-1}\colon\ x_1 + \cdots + x_{d-1} \leqq 1\}$. For an open d station network with fi-

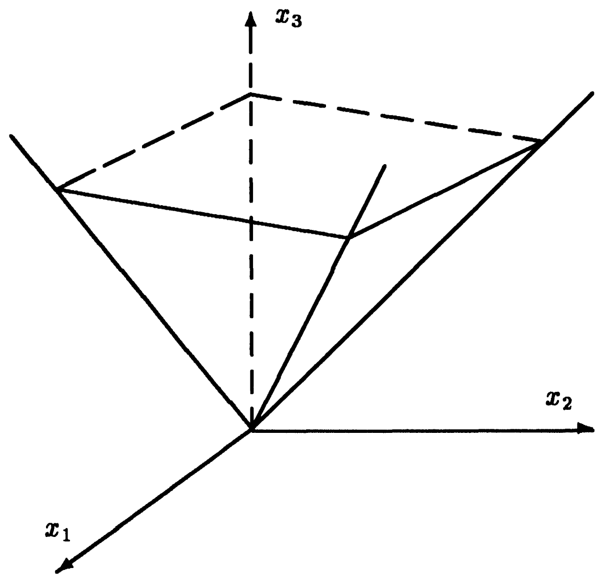

Fig. 1. A nonsimple polyhedron S with $d = 3$, $m = 4$, $n_1 = (1,0,0)'$, $n_2 = (0,1,0)'$, $n_3 = (-1,0,1)'$, $n_4 = (0,-1,1)'$ and $b = (0,0,0,0)'$. The square (with half dashed, half solid line boundary) sketched in the above figure indicates a cross section of the boundary of the (infinite) polyhedral cone S.

nite buffer constraints at each station, the proposed approximating SRBM's live in a bounded simple polyhedron in $\mathbf{R}_+^d$ (see [11]) for a two-dimensional example. Recently, Nguyen [30] studied Brownian models of processing networks for which the approximating SRBM's may live in a nonsimple convex polyhedron such as that pictured in Fig. 1. Heavy traffic limit theorems have been proved to justify the above approximations in some cases and in particular for open [32] and closed [9] single class networks, a multiclass station with feedback [33], [14], feedforward multiclass networks [31], and feedforward fork and join networks [30]. However, it is currently not known in general what multiclass networks with feedback can be approximated by SRBM's under conditions of heavy traffic. Recent work of authors such as Dai and Wang [16], Dai and Nguyen [15], and Whitt [38] suggest that not all multiclass networks with feedback can have such approximations. Indeed, Rybko and Stolyar [35] and Bramson [5], [6] have recently shown that the fundamental question of stability for a nondeterministic multiclass queueing network has not been resolved; in particular, the traditional definition of heavy traffic in terms of nominal traffic intensities being close to one at each station is not appropriate for all multiclass networks (see [38]). It is a challenging open problem to determine a suitable class of multiclass networks with feedback for which there exists a heavy traffic diffusion approximation. The results of this paper are intended to provide a mathematical foundation for the existence and uniqueness of SRBM's, from which point one might try to establish a heavy traffic limit theorem for a suitable class of multiclass networks and also proceed to develop further analysis of the SRBM's. In fact, the latter has already begun. Recently, Dai and Kurtz [13] used our results to establish a characterization for the stationary distributions of SRBM's in terms of a *basic adjoint relationship*. This relationship is the starting point for a numerical algorithm proposed by Dai and Harrison [11] for the computation of stationary distributions of SRBM's.

We conclude this section with some notational conventions used throughout this work. Consider a closed set F in $\mathbf{R}^d$. Let $C(F)$ denote the collection of all continuous real valued functions defined on F. Let $C_b(F)$ denote the collection of all functions in $C(F)$ that are bounded on F. Define the norm on functions $f \in C_b(F)$ by $\|f\|_\infty = \sup_{x\in F}|f(x)|$. For $n = 1, 2, \ldots, \infty$, let $C^n(F)$ denote the collection of all functions $f\colon F \to \mathbf{R}$ that can be extended to be n-times continuously differentiable on some domain containing F. The symbol $C_b^n(F)$ will denote the collection of all functions in $C^n(F)$ that together with their derivatives up to and including those of order n (if $n < \infty$) are bounded on F. For any function $f \in C^2(F)$, define on F,

$$Lf = \frac{1}{2}\sum_{i,j=1}^{d} \Gamma_{ij}\,\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\theta_i \frac{\partial f}{\partial x_i}, \tag{1.5}$$

$$D_i f = v_i \cdot \nabla f, \qquad i = 1, \ldots, m, \tag{1.6}$$

where ∇f denotes the gradient of f. For $x \in \mathbf{R}^d$ and $r > 0$, let $|x|$ denote the Euclidean norm of x, δ_x denote the unit probability mass at the point x, $B(x, r)$ denote the open ball with center x and radius r in $\mathbf{R}^d$, and $d(x, G)$ denote the distance from x to the closure of $G \subset \mathbf{R}^d$. For any metric space A, let $D_A[0, \infty)$ denote the space of functions from $[0, \infty)$ into A that are right continuous on $[0, \infty)$ and have finite left limits on $(0, \infty)$. We endow $D_A[0, \infty)$ with the Skorokhod topology (cf. [17, § 3.5]). The subset of $D_A[0, \infty)$ consisting of all *continuous* functions from $[0, \infty)$ into A is denoted by $C_A[0, \infty)$ and has the topology induced from $D_A[0, \infty)$.

2. Patchwork martingale problem. The following definition of a patchwork martingale problem is a very slight adaptation to our situation of the notion introduced by Kurtz in [28].

DEFINITION 2.1 (patchwork martingale problem). For $x \in S$, a solution of the *patchwork martingale problem* for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x is an $\{\mathcal{F}_t\}$-adapted, $(d+m+1)$-dimensional process $(\xi, \lambda_0, \ldots, \lambda_m)$ defined on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ such that ξ is a d-dimensional process, $\lambda = (\lambda_0, \ldots, \lambda_m)'$ is an $(m+1)$-dimensional process, and

(i) ξ has continuous paths in S, P_x-a.s.,

(ii) $\xi(0) = x$, P_x-a.s.,

(iii) P_x-a.s.,

(a) $\lambda_i(0) = 0$, $i = 0, 1, \ldots, m$,

(b) λ_i is continuous and nondecreasing, $i = 0, 1, \ldots, m$,

(c) λ_i can increase only when ξ is on F_i, i.e., $\int_0^t 1_{F_i}(\xi(s))\,d\lambda_i(s) = \lambda_i(t)$ for all $t \geqq 0$, $i = 1, \ldots, m$,

(d) $\sum_{i=0}^m \lambda_i(t) = t$ for all $t \geqq 0$,

(iv) under P_x,

$$f(\xi(t)) - \int_0^t Lf(\xi(s))\,d\lambda_0(s) - \sum_{i=1}^m \int_0^t D_i f(\xi(s))\,d\lambda_i(s), \qquad t \geqq 0, \tag{2.1}$$

is an $\{\mathcal{F}_t\}$-martingale for each $f \in C_b^2(S)$.

Remark. The process $(\xi, \lambda_0, \ldots, \lambda_m)$ (or when convenient, ξ) will be called a solution of the patchwork martingale problem for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x when the accompanying probability space on which it is defined is clear from the context.

THEOREM 2.1. *For each $x \in S$, there exists a solution of the patchwork martingale problem for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x.*

Our proof of Theorem 2.1 follows the general outline of the proof of Lemma 6.1.1 in [28]. However, a variety of things need to be specifically verified for our case, so we give full details of the proof here. We first establish some preliminary lemmas.

Since Γ is symmetric and positive definite, there is a (symmetric) matrix A such that $\Gamma = AA'$. Let $\alpha_1, \ldots, \alpha_d$ be the d columns of A. For each integer $n \geqq 1$, $f \in C_b^2(\mathbf{R}^d)$, and $x \in \mathbf{R}^d$, we define a discrete approximation to $Lf(x)$ via

$$\begin{aligned} L^n f(x) &= \frac{n}{4d} \sum_{i=1}^{d} \left(f\left(x + \left(\frac{2d}{n}\right)^{1/2} \alpha_i \right) + f\left(x - \left(\frac{2d}{n}\right)^{1/2} \alpha_i \right) - 2f(x) \right) \\ &\quad + \frac{n}{2} \left(f\left(x + \left(\frac{2}{n}\right)\theta \right) - f(x) \right) \equiv n \int_{\mathbf{R}^d} (f(y) - f(x))\, \mu_L^n(x, dy), \end{aligned} \tag{2.2}$$

where

$$\mu_L^n(x, \cdot) = \frac{1}{4d} \sum_{i=1}^{d} \left(\delta_{x+(2d/n)^{1/2}\alpha_i} + \delta_{x-(2d/n)^{1/2}\alpha_i} \right) + \frac{1}{2} \delta_{x+(2/n)\theta}.$$

For each maximal $\mathbf{K} \subset \mathbf{J}$, by condition **(S.a)**, there exist real numbers $a_i^{\mathbf{K}} > 0$, $i \in \mathbf{K}$, such that for $v_{\mathbf{K}} \equiv \sum_{i \in \mathbf{K}} a_i^{\mathbf{K}} v_i$, $n_i \cdot v_{\mathbf{K}} > 0$ for all $i \in \mathbf{K}$. Without loss of generality, we suppose that $\sum_{i \in \mathbf{K}} a_i^{\mathbf{K}} = 1$. For $f \in C_b^2(\mathbf{R}^d)$, define

$$D_{\mathbf{K}} f = v_{\mathbf{K}} \cdot \nabla f.$$

For each integer $n \geqq 1$, $f \in C_b^2(\mathbf{R}^d)$, and $x \in \mathbf{R}^d$, define a discrete approximation to $D_{\mathbf{K}}$ as follows:

$$D_{\mathbf{K}}^n f(x) = n \left(f\left(x + \frac{1}{n} v_{\mathbf{K}} \right) - f(x) \right) = n \int_{\mathbf{R}^d} (f(y) - f(x))\, \mu_{\mathbf{K}}^n(x, dy), \tag{2.3}$$

where $\mu_{\mathbf{K}}^n(x, \cdot) = \delta_{x+(1/n)v_{\mathbf{K}}}$.

LEMMA 2.2. *Let $f \in C_b^2(\mathbf{R}^d)$ and $n \geqq 1$ be an integer. Then*

$$\|L^n f\|_\infty \leqq \frac{1}{2} \sum_{i=1}^{d} \|\alpha_i \cdot \nabla(\alpha_i \cdot \nabla f)\|_\infty + \|\theta \cdot \nabla f\|_\infty, \tag{2.4}$$

and for each maximal $\mathbf{K} \subset \mathbf{J}$,

$$\|D_{\mathbf{K}}^n f\|_\infty \leqq \|D_{\mathbf{K}} f\|_\infty. \tag{2.5}$$

Furthermore,

$$L^n f \to Lf \quad \text{as} \quad n \to \infty, \tag{2.6}$$

and for each maximal $\mathbf{K} \subset \mathbf{J}$,

$$D_{\mathbf{K}}^n f \to D_{\mathbf{K}} f \quad \text{as} \quad n \to \infty, \tag{2.7}$$

where the convergence in (2.6)–(2.7) is uniform on each compact set in $\mathbf{R}^d$.

Proof. The claims follow easily from the following representations which hold for all $x \in \mathbf{R}^d$:

$$L^n f(x) = \frac{1}{2} \int_0^1 \int_{-1}^1 t \sum_{i=1}^d \alpha_i \cdot \nabla(\alpha_i \cdot \nabla f)\left(x + st\left(\frac{2d}{n}\right)^{1/2} \alpha_i\right) ds\, dt$$

$$+ \int_0^1 \theta \cdot \nabla f\left(x + t\left(\frac{2}{n}\right)\theta\right) dt, \tag{2.8}$$

$$D^n_{\mathbf{K}} f(x) = \int_0^1 D_{\mathbf{K}} f\left(x + t\left(\frac{1}{n}\right) v_{\mathbf{K}}\right) dt. \tag{2.9}$$

For a point $x \in S$, in general we do not have that the supports of the probability measures $\mu^n_L(x, \cdot)$ and $\mu^n_{\mathbf{K}}(x, \cdot)$ are contained in S. Accordingly, we need to slightly enlarge the state space S as follows. For each $n \geqq 1$, define

$$c_n = \left(\frac{2d}{n}\right)^{1/2} \max_{1 \leqq i \leqq d} |\alpha_i|, \qquad S_n = \left\{x \in \mathbf{R}^d \colon n_i \cdot x - b_i \geqq -c_n,\ \forall\, i \in \mathbf{J}\right\}. \tag{2.10}$$

For any $\mathbf{K} \subset \mathbf{J}$, $\mathbf{K} \neq \varnothing$, define $F^n_{\mathbf{K}} \equiv \{x \in S_n \colon n_i \cdot x \leqq b_i,\ \ \forall\, i \in \mathbf{K}$ and $n_i \cdot x > b_i,\ \ \forall\, i \in \mathbf{J} \backslash \mathbf{K}\}$. Let S^0 denote the interior of S. We make the following convention to allow $\mathbf{K} = \varnothing$: $F_\varnothing = S$, $F^n_\varnothing = S^0$, $D_\varnothing = L$, $D^n_\varnothing = L^n$ and $\mu^n_\varnothing = \mu^n_L$. In addition, for a nonmaximal $\mathbf{K} \subset \mathbf{J}$ for which $\mathbf{K} \neq \varnothing$ and $F_{\mathbf{K}} \neq \varnothing$, there is a unique maximal $\mathbf{L} \subset \mathbf{J}$ such that $\mathbf{K} \subset \mathbf{L}$ and $F_{\mathbf{K}} = F_{\mathbf{L}}$. In this case we define $v_{\mathbf{K}} = v_{\mathbf{L}}$, $D_{\mathbf{K}} = D_{\mathbf{L}}$, $D^n_{\mathbf{K}} = D^n_{\mathbf{L}}$, and $\mu^n_{\mathbf{K}} = \mu^n_{\mathbf{L}}$. If $\mathbf{K} \neq \varnothing$ and $F_{\mathbf{K}} = \varnothing$, we define $v_{\mathbf{K}} = 0$, $D_{\mathbf{K}} \equiv 0$, $D^n_{\mathbf{K}} \equiv 0$, and $\mu^n_{\mathbf{K}}(x, \cdot) = \delta_x$ for all $x \in \mathbf{R}^d$. In the following, note that $\mathbf{K} \subset \mathbf{J}$ includes the case where $\mathbf{K} = \varnothing$.

Now,

$$S_n = \bigcup_{\mathbf{K} \subset \mathbf{J}} F^n_{\mathbf{K}}, \tag{2.11}$$

where the nonempty sets in the union are disjoint. Let n_0 be such that for $n \geqq n_0$,

$$\frac{2}{n}|\theta| \leqq c_n \quad \text{and} \quad \frac{1}{n}|v_{\mathbf{K}}| \leqq c_n \qquad \forall\, \mathbf{K} \colon \varnothing \neq \mathbf{K} \subset \mathbf{J}. \tag{2.12}$$

Then we have the following.

LEMMA 2.3. *Let $n \geqq n_0$ be fixed. For each $\mathbf{K} \subset \mathbf{J}$ and each $x \in F^n_{\mathbf{K}}$, $\mu^n_{\mathbf{K}}(x, \cdot)$ is a probability measure on S_n.*

Proof. First consider $\mathbf{K} = \varnothing$ and $x \in F^n_\varnothing = S^0$. For each $j \in \{1, \ldots, d\}$ and $i \in \mathbf{J}$,

$$n_i \cdot \left(x \pm \left(\frac{2d}{n}\right)^{1/2} \alpha_j\right) = n_i \cdot x \pm \left(\frac{2d}{n}\right)^{1/2} n_i \cdot \alpha_j \geqq n_i \cdot x - \left(\frac{2d}{n}\right)^{1/2} |\alpha_j|$$

$$\geqq b_i - \left(\frac{2d}{n}\right)^{1/2} \max_{1 \leqq j \leqq d} |\alpha_j| = b_i - c_n.$$

Hence $x \pm (2d/n)^{1/2}\alpha_j \in S_n$ for all $j \in \{1, \ldots, d\}$. Similarly, by (2.12), $x + (2/n)\theta \in S_n$. The result for $\mathbf{K} = \varnothing$ then follows from the definition of $\mu^n_\varnothing = \mu^n_L$.

Next, suppose that $\mathbf{K} \subset \mathbf{J}$ such that $\mathbf{K} \neq \varnothing$ and $F_{\mathbf{K}} \neq \varnothing$. Let $\mathbf{L}$ be the unique maximal set such that $\mathbf{K} \subset \mathbf{L}$ and $F_{\mathbf{K}} = F_{\mathbf{L}}$. Fix $x \in F^n_{\mathbf{K}}$. For $i \in \mathbf{K} \subset \mathbf{L}$,

$$n_i \cdot \left(x + \frac{1}{n} v_{\mathbf{L}}\right) = n_i \cdot x + \frac{1}{n} n_i v_{\mathbf{L}} \geqq n_i \cdot x \geqq b_i - c_n,$$

and for $i \in \mathbf{J}\backslash\mathbf{K}$,

$$n_i \cdot \left(x + \frac{1}{n} v_{\mathbf{L}} \right) = n_i \cdot x + \frac{1}{n} n_i v_{\mathbf{L}} \geqq b_i - \frac{1}{n} |v_{\mathbf{L}}| \geqq b_i - c_n.$$

Thus $x + (1/n)v_{\mathbf{L}} \in S_n$. The result for $\mathbf{K}$ as described above then follows from the definition of $\mu^n_{\mathbf{K}} = \mu^n_{\mathbf{L}}$.

Finally, suppose that $\mathbf{K} \neq \varnothing$, $F_{\mathbf{K}} = \varnothing$. Then for $x \in F^n_{\mathbf{K}}$, $\mu_{\mathbf{K}}(x, \cdot) = \delta_x$ is a probability measure on S_n.

LEMMA 2.4. *Suppose $\xi^n \to \xi$ in $D_{\mathbf{R}^d}[0,\infty)$ and $\lambda^n \to \lambda$ in $C_{\mathbf{R}_+}[0,\infty)$ as $n \to \infty$. Assume λ^n is nondecreasing for each n. Then for any $f \in C_b(\mathbf{R}^d)$,*

$$\int_0^t f(\xi^n(s))\, d\lambda^n(s) \to \int_0^t f(\xi(s))\, d\lambda(s) \qquad \text{as} \quad n \to \infty,$$

uniformly for t in any compact subset of $[0,\infty)$.

Proof. Since $\xi^n \to \xi$ in the Skorokhod topology, there exists a sequence $\{\gamma_n\}$ of continuous, strictly increasing functions mapping $[0,\infty)$ onto $[0,\infty)$ such that $\xi^n \circ \gamma_n(t) \to \xi(t)$ and $\gamma_n(t) \to t$ as $n \to \infty$, uniformly for t in each compact interval (see Proposition 3.5.3 and Remark 3.5.4 of [17]). Fix $t > 0$ and observe that for all $u \in [0,t]$,

$$\begin{aligned} &\int_0^u f(\xi^n(s))\, d\lambda^n(s) - \int_0^u f(\xi(s))\, d\lambda(s) \\ &\quad = \int_0^{\gamma_n^{-1}(u)} \left(f(\xi^n \circ \gamma_n(s)) - f(\xi(s)) \right) d(\lambda^n \circ \gamma_n)(s) \\ &\qquad + \int_u^{\gamma_n^{-1}(u)} f(\xi(s)) d(\lambda^n \circ \gamma_n)(s) + \int_0^u f(\xi(s)) d(\lambda^n \circ \gamma_n - \lambda)(s). \end{aligned} \tag{2.13}$$

The first term in the right member of (2.13) converges to zero as $n \to \infty$, uniformly for all $u \in [0,t]$, because its absolute value is dominated by

$$\max_{0 \leqq s \leqq \gamma_n^{-1}(t)} \left| f(\xi^n \circ \gamma_n(s)) - f(\xi(s)) \right| \lambda^n(t)$$

and f is continuous. The second term is dominated by $\|f\|_\infty \sup_{0 \leqq u \leqq t} |\lambda^n(u) - (\lambda^n \circ \gamma_n)(u)|$, which also converges to zero as $n \to \infty$. To treat the third term, note that since $f(\xi(\cdot)) \in D_{\mathbf{R}}[0,\infty)$, by Theorem 3.5.6, Proposition 3.5.3, and Remark 3.5.4 of [17], there is a sequence of step functions $\{z^k\}_{k=1}^\infty$ of the form

$$z^k(\cdot) = \sum_{i=1}^{l_k} z^k(t_i^k) 1_{[t_i^k, t_{i+1}^k)}(\cdot),$$

where $0 = t_1^k < t_2^k < \cdots < t_{l_k+1}^k < \infty$ and $\sup_{0 \leqq s \leqq t} |f(\xi(s)) - z^k(s)| \to 0$ as $k \to \infty$. Then,

$$\begin{aligned} \left| \int_0^u f(\xi(s))\, d(\lambda^n \circ \gamma_n - \lambda)(s) \right| &\leqq \left| \int_0^u \left(f(\xi(s)) - z^k(s) \right) d(\lambda^n \circ \gamma_n - \lambda)(s) \right| \\ + \left| \int_0^u z^k(s)\, d(\lambda^n \circ \gamma_n - \lambda)(s) \right| &\leqq \sup_{0 \leqq s \leqq t} \left| f(\xi(s)) - z^k(s) \right| (\lambda^n \circ \gamma_n(t) + \lambda(t)) \\ + \sup_{0 \leqq u \leqq t} \sum_{i=1}^{l_k} \left| z^k(t_i^k \wedge u) \right| &\left| (\lambda^n \circ \gamma_n - \lambda)(t_{i+1}^k \wedge u) - (\lambda^n \circ \gamma_n - \lambda)(t_i^k \wedge u) \right|. \end{aligned}$$

For fixed k, the last term in the above can be made as small as we like for all n sufficiently large. The desired result then follows.

Proof of Theorem 2.1. Let $n \geqq n_0$ be fixed. For any $x \in S_n$ define

$$\mu^n(x, \cdot) = \sum_{\mathbf{K} \subset \mathbf{J}} 1_{F^n_{\mathbf{K}}}(x) \mu^n_{\mathbf{K}}(x, \cdot).$$

Then by (2.11) and Lemma 2.3, for each $x \in S_n$, $\mu^n(x, \cdot)$ is a probability measure on S_n and hence μ^n is a one-step probability transition function for a Markov chain on S_n. Fix $x \in S$ and let $\{\eta^n(k),\ k \geqq 0\}$ be a realization of this Markov chain with starting point x. Let $\xi^n(t) = \eta^n([nt])$ and $\mathcal{F}^{\xi^n}_t = \sigma\{\xi^n(s)\colon 0 \leqq s \leqq t\}$ for all $t \geqq 0$, where $[nt]$ denotes the integer part of nt. For each $\mathbf{K} \subset \mathbf{J}$, define

$$\lambda^n_{\mathbf{K}}(t) = \int_0^t 1_{F^n_{\mathbf{K}}}(\xi^n(s))\, ds, \qquad \text{for all} \quad t \geqq 0.$$

Then, by the form of the infinitesimal generator for η^n and since η^n has bounded jumps (cf. [17, pp. 158, 162]), we have for any $f \in C^2(\mathbf{R}^d)$ that

$$\begin{aligned} f(\xi^n(t)) &- \sum_{k=0}^{[nt]-1} \int_{S_n} \left(f(y) - f(\eta^n(k))\right) \mu^n(\eta^n(k), dy) \\ &= f(\xi^n(t)) - \int_0^{[nt]/n} \sum_{\mathbf{K} \subset \mathbf{J}} 1_{F^n_{\mathbf{K}}}(\xi^n(s))\, D^n_{\mathbf{K}} f(\xi^n(s))\, ds \\ &= f(\xi^n(t)) - \sum_{\mathbf{K} \subset \mathbf{J}} \int_0^{[nt]/n} D^n_{\mathbf{K}} f(\xi^n(s))\, d\lambda^n_{\mathbf{K}}(s) \end{aligned} \tag{2.14}$$

is an $\{\mathcal{F}^{\xi^n}_t\}$-martingale.

We wish to establish tightness of the probability measures induced on $D_{S_{n_0}}[0, \infty)$ by $\{\xi^n\colon n \geqq n_0\}$. For this, we first verify a compact containment property. Let $f(y) = |y|^2$ for all $y \in \mathbf{R}^d$. For $\mathbf{K} \neq \varnothing$, $F_{\mathbf{K}} = \varnothing$, we have $D^n_{\mathbf{K}} f \equiv 0$. For $\mathbf{K} \neq \varnothing$, $F_{\mathbf{K}} \neq \varnothing$, and $\mathbf{L}$ maximal such that $\mathbf{K} \subset \mathbf{L}$ and $F_{\mathbf{K}} = F_{\mathbf{L}}$, we have for $y \in \mathbf{R}^d$,

$$D^n_{\mathbf{K}} f(y) = n\left(\left|y + \frac{1}{n} v_{\mathbf{L}}\right|^2 - |y|^2\right) = 2 v_{\mathbf{L}} \cdot y + \frac{1}{n} |v_{\mathbf{L}}|^2.$$

For $\mathbf{K} = \varnothing$, $y \in \mathbf{R}^d$,

$$\begin{aligned} D^n_{\mathbf{K}} f(y) = \frac{n}{4d} \sum_{i=1}^d &\left(\left|y + \left(\frac{2d}{n}\right)^{1/2} \alpha_i\right|^2 + \left|y - \left(\frac{2d}{n}\right)^{1/2} \alpha_i\right|^2 - 2|y|^2 \right) \\ &+ \frac{n}{2}\left(\left|y + \frac{2}{n}\theta\right|^2 - |y|^2\right) = \sum_{i=1}^d |\alpha_i|^2 + 2\theta \cdot y + \frac{2}{n}|\theta|^2. \end{aligned} \tag{2.15}$$

Let $M > |x|$ and $\tau^n_M = \inf\{t \geqq 0\colon |\xi^n(t)| \geqq M\}$. Then by stopping the martingale in (2.14) at τ^n_M and taking expectations, we deduce that for fixed $t > 0$,

$$\begin{aligned} \mathbf{E}\left[f(\xi^n(t \wedge \tau^n_M))\right] &= |x|^2 + \mathbf{E}\left(\sum_{\mathbf{K} \subset \mathbf{J}} \int_0^{[n(t \wedge \tau^n_M)]/n} D^n_{\mathbf{K}} f(\xi^n(s))\, d\lambda^n_{\mathbf{K}}(s) \right) \\ &\leqq |x|^2 + t\left(\max_{\mathbf{L}} \left(2|v_{\mathbf{L}}|M + \frac{1}{n}|v_{\mathbf{L}}|^2\right) + \sum_{i=1}^d |\alpha_i|^2 + 2|\theta|M + \frac{2}{n}|\theta|^2 \right), \end{aligned}$$

where the $\max_{\mathbf{L}}$ is over all maximal $\mathbf{L} \subset \mathbf{J}$. Thus, taking into account the definition of f, we deduce that

$$M^2 \mathbf{P}(\tau_M^n \leqq t) \leqq |x|^2 + tC(M+1), \tag{2.16}$$

where C is a constant that does not depend on n, t or M. The compact containment property (3.9.1) of [17] follows easily from (2.16).

We now verify that the other conditions for tightness described in Theorem 3.9.1 of [17] hold. Note that the collection Λ of functions formed by restricting functions in $C_b^2(\mathbf{R}^d)$ to S_{n_0} is a dense algebra in $C_b(S_{n_0})$. Now for any fixed $f \in C_b^2(\mathbf{R}^d)$ and $\mathbf{K} \subset \mathbf{J}$, by Lemma 2.2, $||D_{\mathbf{K}}^n f||_\infty$ is bounded uniformly with respect to n. It follows from this and the martingale property of (2.14) that the conditions of Theorem 3.9.4 and Remark 3.9.5(b) of [17] are satisfied for all functions $f \in \Lambda$.

It follows from the above and Theorem 3.9.1 of [17], that the probability measures induced on $D_{S_{n_0}}[0,\infty)$ by $\{\xi^n\colon\ n \geqq n_0\}$ are tight. Also note that for any $\mathbf{K} \subset \mathbf{J}$, $\lambda_{\mathbf{K}}^n$ is Lipschitz continuous with derivative bounded by one. Hence the probability measures induced on $C_{\mathbf{R}_+}[0,\infty)$ by $\{\lambda_{\mathbf{K}}^n\colon\ n \geqq n_0\}$ form a tight sequence. Therefore, the probability measures induced on $D_{S_{n_0}}[0,\infty) \times \prod_{\mathbf{K}\subset\mathbf{J}} C_{\mathbf{R}_+}[0,\infty)$ by $\{(\xi^n; \lambda_{\mathbf{K}}^n,\ \mathbf{K} \subset \mathbf{J})\colon\ n \geqq n_0\}$ form a tight sequence. (Note that if the $\lambda_{\mathbf{K}}^n$ were only known to be in $D_{\mathbf{R}_+}[0,\infty)$, this result would not be true. We rely on the fact that the $\lambda_{\mathbf{K}}^n$ are continuous here to deduce the joint tightness, given the individual tightness of the measures associated with ξ^n and the $\lambda_{\mathbf{K}}^n$.) It follows that there exists a subsequence of $\{(\xi^n; \lambda_{\mathbf{K}}^n, \mathbf{K} \subset \mathbf{J})\colon\ n \geqq n_0\}$ which converges in distribution to a process $(\xi; \lambda_{\mathbf{K}}, \mathbf{K} \subset \mathbf{J})$ with $\xi \in D_{S_{n_0}}[0,\infty)$ and $\lambda_{\mathbf{K}} \in C_{\mathbf{R}_+}[0,\infty)$ for each $\mathbf{K} \subset \mathbf{J}$. Without loss of generality we assume that the original sequence is convergent, and by the Skorokhod representation theorem (cf. [17, Theorem 3.1.8]) that almost surely as $n \to \infty$, $\xi^n \to \xi$ in $D_{S_{n_0}}[0,\infty)$ and $\lambda_{\mathbf{K}}^n \to \lambda_{\mathbf{K}}$ in $C_{\mathbf{R}_+}[0,\infty)$ for all $\mathbf{K} \subset \mathbf{J}$.

Fix $t > 0$ and $\mathbf{K} \subset \mathbf{J}$. Define random measures $\{\nu_{\mathbf{K}}^n\}$ and $\nu_{\mathbf{K}}$ by

$$\nu_{\mathbf{K}}^n(B) = \int_0^t 1_B\big(\xi^n(s)\big)\, d\lambda_{\mathbf{K}}^n(s) \quad \text{and} \quad \nu_{\mathbf{K}}(B) = \int_0^t 1_B\big(\xi(s)\big)\, d\lambda_{\mathbf{K}}(s),$$

for all Borel sets $B \subset \mathbf{R}^d$. By the almost sure convergence assumed at the end of the last paragraph and Lemma 2.4, almost surely the sequence of measures $\{\nu_{\mathbf{K}}^n\}$ converges weakly to $\nu_{\mathbf{K}}$ as $n \to \infty$. Since $\nu_{\mathbf{K}}^n$ is almost surely supported on $F_{\mathbf{K}}^n$, it follows that almost surely $\nu_{\mathbf{K}}$ is supported on $F_{\mathbf{K}}$. For $f \in C_b^2(\mathbf{R}^d)$ and $u \in [0,t]$,

$$\begin{aligned}
&\left|\int_0^u D_{\mathbf{K}}^n f\big(\xi^n(s)\big)\, d\lambda_{\mathbf{K}}^n(s) - \int_0^u D_{\mathbf{K}} f\big(\xi(s)\big)\, d\lambda_{\mathbf{K}}(s)\right| \\
&\quad \leqq \left|\int_0^u D_{\mathbf{K}}^n f\big(\xi^n(s)\big)\, d\lambda_{\mathbf{K}}^n(s) - \int_0^u D_{\mathbf{K}} f\big(\xi^n(s)\big)\, d\lambda_{\mathbf{K}}^n(s)\right| \\
&\quad\quad + \left|\int_0^u D_{\mathbf{K}} f\big(\xi^n(s)\big)\, d\lambda_{\mathbf{K}}^n(s) - \int_0^u D_{\mathbf{K}} f\big(\xi(s)\big)\, d\lambda_{\mathbf{K}}(s)\right|.
\end{aligned}$$

The first term in the last member of the above inequality converges to zero almost surely as $n \to \infty$, uniformly for $u \in [0,t]$. This follows from Lemma 2.2, the fact that $\xi^n \to \xi$ in $D_{\mathbf{R}^d}[0,\infty)$ implies that almost surely $\{\xi^n(s)\colon\ 0 \leqq s \leqq t,\ n \geqq n_0\}$ is bounded, and the fact that $\lambda_{\mathbf{K}}^n(s) \leqq s$ for all $s \geqq 0$. Furthermore, the second term converges to zero almost surely, uniformly for $u \in [0,t]$, by Lemma 2.4. Thus,

$$\left(f(\xi^n_\cdot) - \sum_{\mathbf{K}\subset\mathbf{J}} \int_0^\cdot D^n_{\mathbf{K}} f(\xi^n(s))\, d\lambda^n_{\mathbf{K}}(s);\ \xi^n;\ \lambda^n_{\mathbf{K}},\ \mathbf{K}\subset\mathbf{J} \right)$$

converges a.s. and hence in distribution to the same expression without the n's. Combining this with the martingale property of (2.14), the boundedness of f, the uniform boundedness of $\{D^n_{\mathbf{K}} f\}$, and the fact that $\sum_{\mathbf{K}\subset\mathbf{J}} \lambda_{\mathbf{K}}(s) = s$ a.s., we conclude (cf. [17, p. 362]) that $\{f(\xi(t)) - \sum_{\mathbf{K}\subset\mathbf{J}} \int_0^t D_{\mathbf{K}} f(\xi(s))\, d\lambda_{\mathbf{K}}(s),\ t \geqq 0\}$ is an $\{\mathcal{F}_t\}$-martingale, where $\mathcal{F}_t = \sigma\{(\xi(s); \lambda_{\mathbf{K}}(s), \mathbf{K}\subset\mathbf{J})\colon 0 \leqq s \leqq t\}$.

Define $\lambda_0 = \lambda_\varnothing$. Recall that for $\mathbf{K}$ maximal, $v_{\mathbf{K}} = \sum_{i\in\mathbf{K}} a_i^{\mathbf{K}} v_i$ where $a_i^{\mathbf{K}} > 0$ for all $i \in \mathbf{K}$ and $\sum_{i\in\mathbf{K}} a_i^{\mathbf{K}} = 1$. For any $\mathbf{K} \neq \varnothing$ such that $F_{\mathbf{K}} \neq \varnothing$, there is a unique maximal $\mathbf{L} \supset \mathbf{K}$ such that $F_{\mathbf{K}} = F_{\mathbf{L}}$ and by definition $D_{\mathbf{K}} = D_{\mathbf{L}}$. In the remainder of this proof, $\mathbf{L}$ will always be linked to $\mathbf{K}$ in this way. We now define for each $i \in \mathbf{J}$,

$$\lambda_i = \sum_{\substack{\varnothing\neq\mathbf{K}\subset\mathbf{J} \\ \mathbf{K}:\ i\in\mathbf{L}}} a_i^{\mathbf{L}} \lambda_{\mathbf{K}},$$

where the sum is over all $\mathbf{K}$: $\varnothing \neq \mathbf{K} \subset \mathbf{J}$, and the unique maximal $\mathbf{L}$ corresponding to $\mathbf{K}$ contains i. Note that since $\nu_{\mathbf{K}}$ is supported on $F_{\mathbf{K}}$, there is no contribution to the sum from terms for which $F_{\mathbf{K}} = \varnothing$. It is easily verified from the martingale property mentioned above that for any $f \in C_b^2(\mathbf{R}^d)$,

$$\begin{aligned} f(\xi(t)) &- \int_0^t Lf(\xi(s))\, d\lambda_0(s) - \sum_{i=1}^m \int_0^t D_i f(\xi(s))\, d\lambda_i(s) \\ &= f(\xi(t)) - \sum_{\mathbf{K}\subset\mathbf{J}} \int_0^t D_{\mathbf{K}} f(\xi(s))\, d\lambda_{\mathbf{K}}(s) \end{aligned}$$

is an $\{\mathcal{F}_t\}$-martingale. Thus, (iv) of Definition 2.1 holds because any $f \in C_b^2(S)$ can be extended to a function in $C_b^2(\mathbf{R}^d)$. Since $\sum_{\mathbf{K}\subset\mathbf{J}} \lambda^n_{\mathbf{K}}(t) = t$ for each $n \geqq n_0$, by taking limits we have $\sum_{\mathbf{K}\subset\mathbf{J}} \lambda_{\mathbf{K}}(t) = t$ a.s., which in turn implies $\sum_{i=0}^m \lambda_i(t) = t$ a.s., by the choice of normalization of the $a^{\mathbf{K}}$ for $\mathbf{K}$ maximal. Also, λ_i is continuous, $\{\mathcal{F}_t\}$-adapted and a.s. is nondecreasing and satisfies $\lambda_i(0) = 0$, because these properties hold for each $\lambda_{\mathbf{K}}$. Thus we have verified conditions (iii) (a)–(b), (d) of Definition 2.1. Condition (ii) is obvious. To check condition (iii) (c), note that for $i = 1, \ldots, m$, $\lambda_i(t) = \int_0^t 1_{F_i}(\xi(s))\, d\lambda_i(s)$ a.s., because $F_{\mathbf{K}} \subset F_i$ for each $\mathbf{K}$ with $i \in \mathbf{L}$ and $\nu_{\mathbf{K}}$ is almost surely supported on $F_{\mathbf{K}}$. The fact that the paths of ξ are a.s. in S holds because $\xi^n \in D_{S_n}[0,\infty)$ for each n and so $\xi \in D_S[0,\infty)$ a.s. It remains to show that ξ is a.s. continuous. We shall use the result in Appendix A for this. Note that by truncating functions in $C^\infty(S)$ to make them bounded outside large compact sets, one can show that (2.1) is an $\{\mathcal{F}_t\}$-local martingale for all $f \in C^\infty(S)$. It is obvious that $C^\infty(S)$ is an algebra, and L and the D_i's map $C^\infty(S)$ into itself. Also L is a derivation as defined in Appendix A and the D_i's satisfy the product rule. For $g(y) = y_i$, (2.1) is a local martingale for $f = g$, g^2, g^3, g^4. By Lemma A.1 of the Appendix we have ξ_i is a.s. continuous for $i = 1, \ldots, d$. Thus all of the properties of Definition 2.1 have been verified.

3. Existence of an SRBM. In this section we prove the existence of an SRBM for given data (S, θ, Γ, R). For this we use a solution to a constrained martingale problem obtained by time-changing a solution of the patchwork martingale problem studied in § 2. The notion of a constrained martingale problem and the idea of obtaining a solution of such by time changing a solution of an associated patchwork martingale problem was introduced by Kurtz in [28]. Note that our definition of a constrained martingale problem is slightly adapted from that in [28].

DEFINITION 3.1 (constrained martingale problem). For $x \in S$, a solution of the *constrained martingale problem* for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x is an $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ such that

(i) Z has continuous paths in S, P_x-a.s.,

(ii) $Z(0) = x$, P_x-a.s.,

(iii) there is an $\{\mathcal{F}_t\}$-adapted, m-dimensional process $Y = (Y_1, \ldots, Y_m)'$ such that for each $i \in \{1, \ldots, m\}$, P_x-a.s.,

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and nondecreasing,

(c) Y_i can increase only when Z is on the face F_i, i.e., $\int_0^t 1_{F_i}(Z(s))\, dY_i(s) = Y_i(t)$ for all $t \geqq 0$,

(iv) under P_x,

$$f(Z(t)) - \int_0^t Lf(Z(s))\, ds - \sum_{i=1}^m \int_0^t D_i f(Z(s))\, dY_i(s), \qquad t \geqq 0, \tag{3.1}$$

is an $\{\mathcal{F}_t\}$-martingale for each $f \in C_b^2(S)$.

Lemmas 3.1 and 3.3 below are key to the time change argument that transforms a solution of the patchwork martingale into one for the constrained martingale problem. For the purpose of these lemmas, let $x \in S$ be fixed and $(\xi, \lambda_0, \ldots, \lambda_m)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ be a solution of the patchwork martingale problem for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x.

LEMMA 3.1. *P_x-a.s., λ_0 is strictly increasing.*

Proof. We first complete $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ and replace $\mathcal{F}_t$ with $\mathcal{F}_{t+}$, so that all P_x-null sets are in the filtration and it is right continuous. Suppose for a proof by contradiction that $P_x(\lambda_0$ is *not* strictly increasing$) > 0$. Then there are rational times $t_0 < t_1$ such that $P_x(\lambda_0(t_1) - \lambda_0(t_0) = 0) > 0$. Let $\tau = \inf\{t > t_0\colon \lambda_0(t) > \lambda_0(t_0)\} - t_0$, and for each $t \geqq 0$ let $\xi^0(t) = \xi(t + t_0)$, $\lambda_i^0(t) = \lambda_i(t_0 + t) - \lambda_i(t_0)$, $i = 0, \ldots, m$. Then by Definition 2.1 (iv), under P_x,

$$f(\xi^0(t)) - \int_0^t Lf(\xi^0(s))\, d\lambda_0^0(s) - \sum_{i=1}^m \int_0^t D_i f(\xi^0(s))\, d\lambda_i^0(s), \qquad t \geqq 0,$$

is an $\{\mathcal{F}_{t_0+t}\}$-martingale for every $f \in C_b^2(S)$. Because τ is an $\{\mathcal{F}_{t_0+t}\}$-stopping time and λ_0^0 does not increase on $[0, \tau]$, by the optional sampling theorem, under P_x,

$$f(\xi^0(t \wedge \tau)) - \sum_{i=1}^m \int_0^{t\wedge\tau} D_i f(\xi^0(s))\, d\lambda_i^0(s), \qquad t \geqq 0, \tag{3.2}$$

is an $\{\mathcal{F}_{t_0+t}\}$-martingale for every $f \in C_b^2(S)$. Furthermore, by Definition 2.1 (iii) (d), P_x-a.s. for all $t \leqq \tau$,

$$\sum_{i=1}^m \lambda_i^0(t) = \sum_{i=0}^m \{\lambda_i(t + t_0) - \lambda_i(t_0)\} = (t + t_0) - t_0 = t\,. \tag{3.3}$$

For each $j \in \{1, \ldots, d\}$, $f(w) \equiv w_j$ for $w \in S$ can be approximated by functions $f_n \in C_b^2(S)$ such that f_n agrees with f on $\{w\colon |w| \leqq n\}$. Then by applying (3.2) to each f_n with time truncated at $\sigma_n = \inf\{s \geqq 0\colon |\xi^0(s)| \geqq n\}$ we see that under P_x,

$$M(t) \equiv \xi^0(t \wedge \tau) - \sum_{i=1}^{m} v_i \lambda_i^0(t \wedge \tau), \qquad t \geqq 0,$$

is an a.s. continuous $\{\mathcal{F}_{t_0+t}\}$-local martingale. Furthermore, it follows from (3.2) and Lemma A.1 in the Appendix that the quadratic variation of M_i is 0 and hence P_x-a.s., $M_i \equiv M_i(0)$ for $i = 1, \ldots, m$. Therefore, we have P_x-a.s.,

$$\xi^0(\cdot \wedge \tau) = \xi^0(0) + \sum_{i=1}^{m} v_i \lambda_i^0(\cdot \wedge \tau). \tag{3.4}$$

From (3.3) and the fact that P_x-a.s., λ_i^0 can increase only when $\xi^0 \in F_i$ $(i = 1, \ldots, m)$, we have P_x-a.s. on $\{\tau > 0\}$,

$$\xi^0(t) \in \partial S \qquad \text{for all} \quad t \in [0, \tau]. \tag{3.5}$$

Now we will show that P_x-a.s. on $\{\tau > 0\}$, ξ^0 will leave the boundary ∂S immediately, i.e., for each $s > 0$, there is $t \in [0, s]$ such that $\xi^0(t) \notin \partial S$. Because $P_x(\tau > 0) > 0$, this will contradict (3.5) and the lemma will be proved.

Let $\omega \in \{\tau > 0\}$ and suppose that Definition 2.1 (iii), (3.3), (3.4), (3.5) hold for this ω. In fact, we will show that if $\mathbf{K}$ is the largest index set such that $\xi^0(0, \omega) \in F_{\mathbf{K}} \equiv \cap_{j \in \mathbf{K}} F_j$, then $\xi^0(\cdot, \omega)$ will leave $F_{\mathbf{K}}$ immediately after time zero. Suppose on the contrary that there is an $s \in (0, \tau(\omega))$ such that

$$\xi^0(t, \omega) \in F_{\mathbf{K}} \qquad \text{for all} \quad t \in [0, s]. \tag{3.6}$$

Note that since $\mathbf{K}$ was chosen to be as large as possible, we may suppose that s is chosen sufficiently small that in addition to (3.6),

$$\xi^0(t, \omega) \notin F_i \qquad \text{for all} \quad t \in [0, s] \quad \text{and} \quad i \in \mathbf{J} \setminus \mathbf{K}. \tag{3.7}$$

From (3.3), (3.7), and the properties of the λ_i's, we have

$$\sum_{j \in \mathbf{K}} \lambda_j^0(t, \omega) = t \qquad \text{for all} \quad t \in [0, s]. \tag{3.8}$$

Since $\mathbf{K}$ must be maximal, by assumption **(S.b)**, there exists a (positive) linear combination η of the unit normals $\{n_j,\ j \in \mathbf{K}\}$ such that $\eta \cdot v_j > 0$ for all $j \in \mathbf{K}$. Then by (3.6), $\eta \cdot (\xi^0(t, \omega) - \xi^0(0, \omega)) = 0$ for all $t \in [0, s]$. From this, (3.4) and (3.6), we have

$$\sum_{j \in \mathbf{K}} \eta \cdot v_j\, \lambda_j^0(t, \omega) = 0 \qquad \text{for all} \quad t \in [0, s]\,, \tag{3.9}$$

which contradicts (3.8). Therefore, $\xi^0(\cdot, \omega)$ leaves $F_{\mathbf{K}}$ immediately. In view of (3.5) and the fact that $\mathbf{K}$ was chosen as large as possible (which implies that $\xi^0(0, \omega)$ is a positive distance from any F_i for $i \notin \mathbf{K}$), there must be a strict subset $\mathbf{K}_1$ of $\mathbf{K}$ and $s_1 \in (0, \tau(\omega))$ such that $\xi^0(s_1, \omega) \in F_{\mathbf{K}_1}$ and $\xi^0(s_1, \omega) \notin F_i$ for any $i \notin \mathbf{K}_1$. By repeating the above argument with $\xi^0(s_1 + \cdot, \omega)$, $\mathbf{K}_1$, in place of $\xi^0(\cdot, \omega)$, $\mathbf{K}$, respectively, we see that $\xi^0(s_1+\cdot,\omega)$ leaves $F_{\mathbf{K}_1}$ immediately. Continuing in this way, through finitely many sets $\mathbf{K}_j$ of decreasing size, we eventually obtain a contradiction to (3.5).

Remark. Part of the proof of this lemma follows the proof of Lemma 6.1.6 of [28]. The main new contribution here is to show that $\xi^0(\cdot,\omega)$ leaves the boundary immediately, which is a condition *assumed* in Kurtz's lemma.

We now state a theorem which plays a key role in this paper.

THEOREM 3.2. *Under conditions* **(S.a)** *and* **(S.b)**, *there is a function* $g \in C_b^2(S)$ *satisfying* $D_i g \geqq 1$ *on* F_i *for all* $i \in \mathbf{J}$.

Proof. The proof is given in Appendix B.

LEMMA 3.3. P_x-*a.s.*, $\lim_{t\to\infty} \lambda_0(t) = +\infty$.

Proof. From Theorem 3.2 we have a function $g \in C_b^2(S)$ such that $D_i g \geqq 1$ on F_i for all $i \in \mathbf{J}$. Therefore, the proof of Lemma 6.1.9 of [28] can be carried over here.

THEOREM 3.4. *For each* $x \in S$, *there exists a solution of the constrained martingale problem for* $(L, S; D_1, F_1; \ldots; D_m, F_m)$ *that starts from* x.

Proof. Fix $x \in S$ and let $(\xi, \lambda_0, \ldots, \lambda_m)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ be a solution of the patchwork martingale problem for $(L, S; D_1, F_1; \ldots; D_m, F_m)$ that starts from x. We assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ has been completed. For each $t \geqq 0$, define the stopping time $\tau(t) = \inf\{s \geqq 0\colon \lambda_0(s) \geqq t\}$. Clearly, $t \to \tau(t)$ is nondecreasing, and by Lemmas 3.1 and 3.3, P_x-a.s., $t \to \tau(t)$ is continuous and real-valued. Letting $\mathcal{F}_\infty = \mathcal{F}$, for each $t \geqq 0$ we define $\mathcal{G}_t = \mathcal{F}_{\tau(t)}$. Then $\{\mathcal{G}_t,\ t \geqq 0\}$ is a filtration. Except on the P_x-null set $\{\tau(s) = +\infty \text{ for some } s\}$, we define for all $t \geqq 0$,

$$Z(t) = \xi\big(\tau(t)\big) \quad \text{and} \quad Y_i(t) = \lambda_i\big(\tau(t)\big) \qquad \text{for} \quad i = 1, \ldots, m,$$

and $Y = (Y_1, \ldots, Y_m)'$. On the exceptional null set, we define $Z \equiv x$ and $Y \equiv 0$. Then, (Z, Y) is P_x-a.s. continuous and it is $\{\mathcal{G}_t\}$-adapted (since we augmented $\{\mathcal{F}_t\}$). Now we show that Z, together with Y, generates a solution of the constrained martingale problem.

First, P_x-a.s., Z has paths in S and $Z(0) = \xi(0) = x$. Fix $i \in \{1, \ldots, m\}$. By composition, P_x-a.s., Y_i is continuous and nondecreasing, and $Y_i(0) = \lambda_i(0) = 0$. Furthermore, P_x-a.s. for each $t \geqq 0$,

$$\begin{aligned}\int_0^t 1_{F_i}\big(Z(s)\big)\, dY_i(s) &= \int_0^t 1_{F_i}\Big(\xi\big(\tau(s)\big)\Big)\, d\lambda_i\big(\tau(s)\big) \\ &= \int_0^{\tau(t)} 1_{F_i}\big(\xi(u)\big)\, d\lambda_i(u) = \lambda_i\big(\tau(t)\big) = Y_i(t).\end{aligned}$$

For $f \in C_b^2(S)$, by (2.1) and the optional stopping theorem, for each positive integer n,

$$\begin{aligned}& f\left(\xi\left(\tau(t) \wedge n\right)\right) - \int_0^{\tau(t)\wedge n} Lf\big(\xi(s)\big)\, d\lambda_0(s) \\ & \qquad - \sum_{i=1}^m \int_0^{\tau(t)\wedge n} D_i f\big(\xi(s)\big)\, d\lambda_i(s), \qquad t \geqq 0,\end{aligned} \tag{3.10}$$

is a $\{\mathcal{G}_t\}$-martingale under P_x. We would like to pass to the limit as $n \to \infty$ and conclude that

$$f\big(Z(t)\big) - \int_0^t Lf\big(Z(s)\big)\, ds - \sum_{i=1}^m \int_0^t D_i f\big(Z(s)\big)\, dY_i(s), \qquad t \geqq 0, \tag{3.11}$$

is a $\{\mathcal{G}_t\}$-martingale under P_x. This is valid if for each fixed t, the expression in (3.10) is uniformly integrable as a sequence of random variables indexed by n. Since f and its first and second derivatives are bounded, and P_x-a.s., $\lambda_0(\tau(t)) = t$ for all $t \geqq 0$ and each of the λ_i is nondecreasing, it suffices for this to show that for each $i \in \{1, \ldots, m\}$,

$$E_x\left[Y_i(t)\right] = E_x\left[\lambda_i\big(\tau(t)\big)\right] < \infty, \tag{3.12}$$

where E_x denotes expectation under P_x. To prove (3.12), we apply (3.10) with f replaced by the function g of Theorem 3.2 and take expectations to conclude that

$$\begin{aligned} &E_x\left[g\left(\xi(\tau(t) \wedge n)\right)\right] - g(x) - E_x\left[\int_0^{\tau(t)\wedge n} Lg\big(\xi(s)\big)\, d\lambda_0(s)\right] \\ &\quad = E_x\left[\sum_{i=1}^m \int_0^{\tau(t)\wedge n} D_i g(\xi(s))\, d\lambda_i(s)\right] \geqq E_x\left[\sum_{i=1}^m \lambda_i\big(\tau(t) \wedge n\big)\right]. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$E_x\left[\sum_{i=1}^m \lambda_i\big(\tau(t)\big)\right] \leqq 2\,\|g\|_\infty + \|Lg\|_\infty\, t < +\infty.$$

Thus, (Z, Y) is a solution of the constrained martingale problem.

THEOREM 3.5. *For each $x \in S$, there exists an SRBM associated with (S, θ, Γ, R) that starts from x.*

Proof. Fix $x \in S$ and let Z with associated process Y be a solution of the constrained martingale problem for $(L, S; D_1, F_1; \ldots; D_m, F_m)$, as described in Definition 3.1. We assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ has been completed. For each $t \geqq 0$, let $W(t) = Z(t) - Z(0) - RY(t) - \theta t$. Then W is an $\{\mathcal{F}_t\}$-adapted, d-dimensional, P_x-a.s. continuous process that starts from the origin.

In the following, the ambient probability measure under which martingales are defined is P_x. We now show that W is a Brownian motion $\{\mathcal{F}_t\}$-martingale with covariance matrix Γ and zero drift. For each integer $n > 0$, let

$$\sigma_n = \inf\left\{t \geqq 0\colon\ \big|Z(t)\big| \geqq n\right\}.$$

Then σ_n is an $\{\mathcal{F}_t\}$-stopping time. Fix $i \in \{1, \ldots, d\}$ and let $f \in C_b^2(S)$ such that $f(w) = w_i$ on $\{w\colon |w| \leqq n\}$. By (3.1) and the optional stopping theorem,

$$f\big(Z(t \wedge \sigma_n)\big) - \int_0^{t\wedge\sigma_n} Lf\big(Z(s)\big)\, ds - \sum_{k=1}^m \int_0^{t\wedge\sigma_n} D_k f\big(Z(s)\big)\, dY_k(s), \qquad t \geqq 0, \tag{3.13}$$

is an a.s. continuous, $\{\mathcal{F}_t\}$-martingale. Observe that for $|w| \leqq n$, $Lf(w) = \theta_i$ and $D_k f(w) = v_{ki}$, where v_{ki} denotes the ith element of the vector v_k. We then see from (3.13) that $W_i(\cdot \wedge \sigma_n)$ is an $\{\mathcal{F}_t\}$-martingale. Since $\sigma_n \uparrow \infty$ as $n \to \infty$, W_i is an $\{\mathcal{F}_t\}$-local martingale. Similarly, by choosing $f \in C_b^2(\mathbf{R}^d)$ such that $f(w) = w_i w_j$ for $|w| \leqq n$, (3.1) gives that

$$\begin{aligned} &Z_i(t \wedge \sigma_n) Z_j(t \wedge \sigma_n) - \Gamma_{ij}(t \wedge \sigma_n) - \int_0^{t\wedge\sigma_n} \big(\theta_i Z_j(s) + \theta_j Z_i(s)\big)\, ds \\ &\qquad - \sum_{k=1}^m \int_0^{t\wedge\sigma_n} \big(v_{ki} Z_j(s) + v_{kj} Z_i(s)\big)\, dY_k(s), \qquad t \geqq 0, \end{aligned} \tag{3.14}$$

is an $\{\mathcal{F}_t\}$-martingale. On the other hand, by Itô's formula, we have P_x-a.s. for all $t \geqq 0$,

$$\begin{aligned} Z_i(t\wedge\sigma_n)Z_j(t\wedge\sigma_n) &= Z_i(0)Z_j(0) + \int_0^{t\wedge\sigma_n} Z_i(s)\,dW_j(s) \\ &+ \int_0^{t\wedge\sigma_n} Z_j(s)\,dW_i(s) + \int_0^{t\wedge\sigma_n} \big(\theta_i Z_j(s) + \theta_j Z_i(s)\big)\,ds + \langle W_i, W_j\rangle(t\wedge\sigma_n) \\ &+ \sum_{k=1}^m \int_0^{t\wedge\sigma_n} \big(v_{ki} Z_j(s) + v_{kj} Z_i(s)\big)\,dY_k(s), \end{aligned} \tag{3.15}$$

where $\langle W_i, W_j\rangle$ is the *mutual variation process* of W_i and W_j. The first two stochastic integrals on the right-hand side of (3.15) define $\{\mathcal{F}_t\}$-local martingales. Then comparing (3.14) with (3.15), we conclude that $\langle W_i, W_j\rangle(t\wedge\sigma_n) - \Gamma_{ij}(t\wedge\sigma_n)$, $t \geqq 0$, is an $\{\mathcal{F}_t\}$-local martingale. Since it is also P_x-a.s. continuous and locally of bounded variation, it follows that it must be constant at its initial value of zero. Letting $n \to \infty$ yields P_x-a.s.,

$$\langle W_i, W_j\rangle(t) = \Gamma_{ij} t \qquad \text{for all} \quad t \geqq 0.$$

It follows that W is a d-dimensional Brownian motion starting from zero with covariance matrix Γ and drift zero. By letting $X(t) = Z(0) + W(t) + \theta t$ for all $t \geqq 0$, we see that X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ, such that $X(0) = x$ P_x-a.s., and $\{X(t) - \theta t, \mathcal{F}_t,\ t \geqq 0\}$ is a martingale. The triple (Z, Y, X) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ satisfies the conditions of Definition 1.1 for an SRBM that starts from x, and so the theorem is proved.

COROLLARY 3.6. *There exists an SRBM associated with* (S, θ, Γ, R).

Proof. By Theorem 3.5, for each $x \in S$ there is an SRBM associated with (S, θ, Γ, R) that starts from x. Let $\{Q_x,\ x \in S\}$ be the probability measures on $(\mathbf{C}, \mathcal{M})$ defined from such solutions via (1.4). It is straightforward to verify that the canonical processes (z, y) together with these probability measures on $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\})$ yield an SRBM, where $X = z - Ry$.

4. State space decomposition, oscillation and boundary behavior. In this section we prove several lemmas that are needed for our proof of uniqueness. These lemmas are also of independent interest for the study of SRBM's.

The following two lemmas will be used several times to localize to subsets of the state space. For this, let C be the constant determined in Lemma B.1 and for each $\varepsilon \geqq 0$ and $\mathbf{K} \subset \mathbf{J}$ (including the empty set), define

$$\begin{aligned} F_{\mathbf{K}}^{\varepsilon} = \{x \in \mathbf{R}^d\colon\ 0 \leqq n_i \cdot x - b_i \leqq C_\varepsilon \ &\text{for all } i \in \mathbf{K} \text{ and } n_i \cdot x - b_i > \varepsilon \\ &\text{for all } i \in \mathbf{J}\backslash\mathbf{K}\}, \end{aligned} \tag{4.1}$$

where $C_\varepsilon = Cm\varepsilon$.

LEMMA 4.1. *For each* $\varepsilon \geqq 0$,

$$S = \bigcup_{\mathbf{K} \in \mathcal{C}} F_{\mathbf{K}}^{\varepsilon}, \tag{4.2}$$

where $\mathcal{C}$ *denotes the collection of subsets of* $\mathbf{J}$ *consisting of all maximal sets in* $\mathbf{J}$ *together with the empty set.*

Proof. Clearly,

$$S = \bigcup_{\mathbf{L} \subset \mathbf{J}} \{x \in \mathbf{R}^d\colon\ 0 \leqq n_i \cdot x - b_i \leqq \varepsilon \ \text{for all } i \in \mathbf{L},$$

$$n_i \cdot x - b_i > \varepsilon \text{ for all } i \in \mathbf{J} \backslash \mathbf{L}\}, \tag{4.3}$$

where $\mathbf{L}$ ranges over all subsets of $\mathbf{J}$, including the empty set. Now, if $\varnothing \neq \mathbf{L} \subset \mathbf{J}$ is not maximal, there is a maximal set $\mathbf{K} \supset \mathbf{L}$ such that $F_{\mathbf{L}} = F_{\mathbf{K}}$. Then, for any $j \in \mathbf{K}$ and $x \in S$, by Lemma B.1,

$$n_j \cdot x - b_j \leqq d(x, F_{\mathbf{K}}) = d(x, F_{\mathbf{L}}) \leqq C \sum_{i \in \mathbf{L}} (n_i \cdot x - b_i)\,. \tag{4.4}$$

Thus,

$$\begin{aligned} &\{x \in \mathbf{R}^d\colon\ 0 \leqq n_i \cdot x - b_i \leqq \varepsilon \text{ for all } i \in \mathbf{L} \text{ and } n_i \cdot x - b_i > \varepsilon \text{ for all } i \in \mathbf{J} \backslash \mathbf{L}\} \\ &\quad \subset \{x \in \mathbf{R}^d\colon\ 0 \leqq n_i \cdot x - b_i \leqq C_\varepsilon \text{ for all } i \in \mathbf{K} \text{ and} \\ &\qquad n_i \cdot x - b_i > \varepsilon \text{ for all } i \in \mathbf{J} \backslash \mathbf{K}\} = F_{\mathbf{K}}^{\varepsilon}. \end{aligned}$$

It follows that the right member of (4.3) is contained in the right member of (4.2). But since the latter is clearly contained in S, the desired equality follows.

For any $\mathbf{L} \subset \mathbf{K} \subset \mathbf{J}$,

$$S^{\mathbf{K}} = \{x \in \mathbf{R}^d\colon\ n_i \cdot x \geqq b_i \text{ for all } i \in \mathbf{K}\}, \tag{4.5}$$

$$F_{\mathbf{L}}^{\mathbf{K}} = \{x \in S^{\mathbf{K}}\colon\ n_i \cdot x = b_i \text{ for all } i \in \mathbf{L}\}. \tag{4.6}$$

and let $R^{\mathbf{K}}$ be the $d \times |\mathbf{K}|$ matrix whose columns are given by $\{v_i, i \in \mathbf{K}\}$.

LEMMA 4.2. *Let* $\mathbf{K} \subset \mathbf{J}$ *be maximal. Then conditions* **(S.a)** *and* **(S.b)** *hold for* $(S^{\mathbf{K}}, R^{\mathbf{K}})$.

Proof. Since conditions **(S.a)** and **(S.b)** hold for (S, R), it suffices to show that if $\mathbf{L}$ is a maximal subset of $\mathbf{K}$, then it is a maximal subset of $\mathbf{J}$. Here $\mathbf{L}$ being maximal as a subset of $\mathbf{K}$ means that $\mathbf{L} \neq \varnothing$, $F_{\mathbf{L}}^{\mathbf{K}} \neq \varnothing$, and $F_{\mathbf{L}}^{\mathbf{K}} \neq F_{\widetilde{\mathbf{L}}}^{\mathbf{K}}$ for any $\widetilde{\mathbf{L}}$ satisfying $\mathbf{L} \subset \widetilde{\mathbf{L}} \subset \mathbf{K}$, $\mathbf{L} \neq \widetilde{\mathbf{L}}$.

Suppose that $\mathbf{L}$ is a maximal subset of $\mathbf{K}$. For a proof by contradiction, suppose that $\mathbf{L}$ is not maximal as a subset of $\mathbf{J}$. Then there is $j \in \mathbf{J} \backslash \mathbf{L}$ such that $F_{\mathbf{L}}^{\mathbf{J}} = F_{\mathbf{L} \cup \{j\}}^{\mathbf{J}}$. We consider the cases where $j \in \mathbf{J} \backslash \mathbf{K}$ and $j \in \mathbf{K} \backslash \mathbf{L}$ separately.

If $j \in \mathbf{J} \backslash \mathbf{K}$, then using the fact that $\mathbf{L} \subset \mathbf{K}$ we have

$$F_{\mathbf{K}}^{\mathbf{J}} = F_{\mathbf{L}}^{\mathbf{J}} \cap F_{\mathbf{K}}^{\mathbf{J}} = F_{\mathbf{L} \cup \{j\}}^{\mathbf{J}} \cap F_{\mathbf{K}}^{\mathbf{J}} = \big\{x \in S\colon\ n_i \cdot x = b_i \text{ for all } i \in \mathbf{K} \cup \{j\}\big\} = F_{\mathbf{K} \cup \{j\}}^{\mathbf{J}},$$

which contradicts the maximality of $\mathbf{K}$ as a subset of $\mathbf{J}$.

On the other hand, if $j \in \mathbf{K} \backslash \mathbf{L}$, then by the maximality of $\mathbf{L}$ as a subset of $\mathbf{K}$, there must be a point $x_1 \in F_{\mathbf{L}}^{\mathbf{K}}$ which is not in $F_{\mathbf{L} \cup \{j\}}^{\mathbf{K}}$. We further claim that there is $x_0 \in F_{\mathbf{K}}^{\mathbf{J}}$ such that

$$\delta_i^0 \equiv n_i \cdot x_0 - b_i > 0 \qquad \text{for all} \quad i \in \mathbf{J} \backslash \mathbf{K}. \tag{4.7}$$

One can see this as follows. First, if $\mathbf{K} = \mathbf{J}$, then since $\mathbf{K}$ is maximal there is an $x_0 \in F_{\mathbf{J}}^{\mathbf{J}}$ and this satisfies (4.7) trivially. Otherwise, if $\mathbf{K} \neq \mathbf{J}$, since $\mathbf{K}$ is maximal, for each $i \in \mathbf{J} \backslash \mathbf{K}$ there is $y_i \in F_{\mathbf{K}}^{\mathbf{J}}$ that is not in $F_{\mathbf{K} \cup \{i\}}^{\mathbf{J}}$. Then let

$$x_0 = \frac{1}{|\mathbf{J} \backslash \mathbf{K}|} \sum_{i \in \mathbf{J} \backslash \mathbf{K}} y_i.$$

It is readily verified that this convex combination of the y_i has the desired properties. Thus in addition to (4.7) we have

$$\begin{aligned} &n_i \cdot x_0 = n_i \cdot x_1 = b_i \qquad \text{for all} \quad i \in \mathbf{L}, \\ &n_i \cdot x_0 = b_i \qquad \text{for all} \quad i \in \mathbf{K}, \qquad n_i \cdot x_1 \geqq b_i \qquad \text{for all} \quad i \in \mathbf{K}, \\ &\quad \delta_j^1 \equiv n_j \cdot x_1 - b_j > 0. \end{aligned}$$

Let $\varepsilon = \min_{i \in \mathbf{J} \setminus \mathbf{K}} \delta_i^0/(|x_1 - x_0| + 1)$, and $x_2 = x_0 + \varepsilon(x_1 - x_0)$. Then, it can be readily verified that $n_i \cdot x_2 = b_i$ for all $i \in \mathbf{L}$, $n_j \cdot x_2 = b_j + \varepsilon\delta_j^1 > b_j$, $n_i \cdot x_2 \geqq b_i$ for all $i \in \mathbf{K}$, $n_i \cdot x_2 = b_i + \delta_i^0 + \varepsilon n_i \cdot (x_1 - x_0) \geqq b_i$ for all $i \in \mathbf{J} \setminus \mathbf{K}$ (by the choice of ε and the fact that $|n_i| = 1$). Thus, $x_2 \in F_{\mathbf{L}}^{\mathbf{J}}$, but $x_2 \notin F_{\mathbf{L} \cup \{j\}}^{\mathbf{J}}$, which contradicts the choice of j.

The following oscillation result is concerned with deterministic continuous paths. For the case $S = \mathbf{R}_+^d$, this result was proved previously by Bernard and El Kharroubi [4]. Our proof of the more general case treated here is adapted from theirs. In the following, for a Borel set $U \subset \mathbf{R}^k$, $k \geqq 1$, we define $C([0,T],U) = \{w\colon [0,T] \to U,\ w \text{ is continuous}\}$.

DEFINITION 4.1. Given $T > 0$ and $x \in C([0,T],\mathbf{R}^d)$ with $x(0) \in S$, an (S,R)-regulation of x over $[0,T]$ is a pair $(z,y) \in C([0,T],\, S) \times C([0,T],\, \mathbf{R}_+^m)$ such that

(i) $z(t) = x(t) + Ry(t)$ for all $t \in [0,T]$,

(ii) $z(t) \in S$ for all $t \in [0,T]$,

(iii) for each $i \in \mathbf{J}$, y_i is nondecreasing, $y_i(0) = 0$, and y_i can increase only at times $t \in [0,T]$ for which $z(t) \in F_i$.

For any continuous function f defined from $[t_1,t_2] \subset [0,\infty)$ into $\mathbf{R}^k$, some $k \geqq 1$, let

$$\operatorname{Osc}\big(f,[t_1,t_2]\big) = \sup_{t_1 \leqq s \leqq t \leqq t_2} \big|f(t) - f(s)\big|.$$

LEMMA 4.3. *There exists a constant κ that depends only on (S,R) such that for any $T > 0$, $x \in C([0,T],\mathbf{R}^d)$ with $x(0) \in S$, and an (S,R)-regulation (y,z) of x over $[0,T]$, the following holds for each interval $[t_1,t_2] \subset [0,T]$:*

$$\operatorname{Osc}\big(y,[t_1,t_2]\big) \leqq \kappa \operatorname{Osc}\big(x,[t_1,t_2]\big) \quad \textit{and} \quad \operatorname{Osc}\big(z,[t_1,t_2]\big) \leqq \kappa \operatorname{Osc}\big(x,[t_1,t_2]\big).$$

Proof. Our proof is adapted from that of Lemma 1 in [4]. It proceeds via an induction on the size of $\mathbf{J}$, the index set for the faces of S. Throughout this proof, T, x, y, z, t_1, t_2 will be as in the statement of the lemma.

First consider the case $|\mathbf{J}| = 1$. Then $R = v_1$ is a vector in $\mathbf{R}^d$ and by **(S.a)**, $n_1 \cdot v_1 > 0$. In this case, y is uniquely given by the one-dimensional regulator mapping for $n_1 \cdot x - b_1$ (cf. [10, Chap. 8]):

$$y(t) = \Big(-\min_{0 \leqq s \leqq t}(n_1 \cdot x - b_1)(s)\Big)^+ \Big/ n_1 \cdot v_1 \qquad \text{for all} \quad t \in [0,T]. \tag{4.8}$$

Together with

$$n_1 \cdot z(t) = n_1 \cdot x(t) + n_1 \cdot v_1 y(t) \qquad \text{for all} \quad t \in [0,T], \tag{4.9}$$

this defines a $([b_1,\infty), n_1 \cdot v_1)$-regulation of $n_1 \cdot x$ over $[0,T]$. The oscillation estimates in the lemma then follow easily from (4.8) and the fact that $z = x + v_1 y$. Thus the lemma holds for $|\mathbf{J}| = 1$.

For the induction step, suppose that the lemma is true for $1 \leqq |\mathbf{J}| < m$. Now consider a state space S with $|\mathbf{J}| = m$. Our proof of the induction step is separated into several parts.

Part a. We claim that there exists a constant C_1 that depends only on (S, R) such that for each $\mathbf{K} \in \mathcal{C}\backslash\{\mathbf{J}\}$ (see Lemma 4.1 for the definition of $\mathcal{C}$), if $y_{\mathbf{J}\backslash\mathbf{K}}$ does not increase on $[t_1, t_2]$, then one has

$$\begin{aligned} &\operatorname{Osc}\big(y, [t_1, t_2]\big) \leqq C_1 \operatorname{Osc}\big(x, [t_1, t_2]\big) \quad \text{and} \\ &\operatorname{Osc}\big(x, [t_1, t_2]\big) \leqq C_1 \operatorname{Osc}\big(x, [t_1, t_2]\big). \end{aligned} \tag{4.10}$$

To see this, note that under the assumptions of the claim, for $t \in [0, t_2 - t_1]$,

$$z(t + t_1) = z(t_1) + x(t + t_1) - x(t_1) + \sum_{i \in \mathbf{K}} v_i\big(y_i(t + t_1) - y_i(t_1)\big). \tag{4.11}$$

It follows that $(z(\cdot + t_1), y_{\mathbf{K}}(\cdot + t_1) - y_{\mathbf{K}}(t_1))$ is an $(S^{\mathbf{K}}, R^{\mathbf{K}})$-regulation of $z(t_1) + x(\cdot + t_1) - x(t_1)$ over $[0, t_2 - t_1]$. If $\mathbf{K} = \varnothing$, then y does not increase on $[t_1, t_2]$ and the oscillation estimate trivially holds with $C_1 = 1$. If $\mathbf{K} \neq \varnothing$, then $\mathbf{K}$ is maximal and so by Lemma 4.2, **(S.a)** and **(S.b)** hold for $(S^{\mathbf{K}}, R^{\mathbf{K}})$. Then, by the induction assumption, since $|\mathbf{K}| < m$, we have that there exists a constant $C_{\mathbf{K}}$ that depends only on $(S^{\mathbf{K}}, R^{\mathbf{K}})$, such that

$$\begin{aligned} \operatorname{Osc}\big(z, [t_1, t_2]\big) &= \operatorname{Osc}\big(z(\cdot + t_1), [0, t_2 - t_1]\big) && (4.12)\\ &\leqq C_{\mathbf{K}} \operatorname{Osc}\big(x(\cdot + t_1) - x(t_1) + z(t_1), [0, t_2 - t_1]\big) && (4.13)\\ &= C_{\mathbf{K}} \operatorname{Osc}\big(x, [t_1, t_2]\big), && (4.14) \end{aligned}$$

and similarly,

$$\operatorname{Osc}\big(y, [t_1, t_2]\big) = \operatorname{Osc}\big(y_{\mathbf{K}}, [t_1, t_2]\big) \leqq C_{\mathbf{K}} \operatorname{Osc}\big(x, [t_1, t_2]\big).$$

The claim then follows by taking C_1 to be the maximum of the $C_{\mathbf{K}}$'s for $\mathbf{K}$ running through $\mathcal{C}\backslash\{\mathbf{J}\}$.

For Parts (b) and (c), we let $\varepsilon = \operatorname{Osc}(x, [t_1, t_2])$. Without loss of generality we assume that $\varepsilon > 0$. By Lemma 4.1, $z(t_1) \in F_{\mathbf{K}}^{C_1\varepsilon}$ for some $\mathbf{K} \in \mathcal{C}$.

Part b. Suppose that the $\mathbf{K}$ found above is not $\mathbf{J}$. Then, for all $i \in \mathbf{J}\backslash\mathbf{K}$, $d(z(t_1), F_i) \geqq n_i \cdot z(t_1) - b_i > C_1\varepsilon$. Applying the result in part (a) to intervals $[t_1, t_2']$ with $t_2' \leqq t_2$ shows that z does not reach F_i for any $i \in \mathbf{J}\backslash\mathbf{K}$ during the interval $[t_1, t_2]$ and therefore $y_{\mathbf{J}\backslash\mathbf{K}}$ does not increase on $[t_1, t_2]$. Then Part (a) implies that (4.10) holds in this case also.

Part c. Suppose that the $\mathbf{K}$ described before part (b) is equal to $\mathbf{J}$. Since $z(t_1) \in F_{\mathbf{J}}^{C_1\varepsilon}$, by Lemma B.1, $d(z(t_1), F_i) \leqq C_2\varepsilon$, where $C_2 = C_1 C m$. Now one of the following two situations holds.

(i) For every $i \in \mathbf{J}$, $d(z(t), F_i) \leqq 2C_2\,\varepsilon$ for all $t \in [t_1, t_2]$.

Then for each $i \in \mathbf{J}$,

$$0 \leqq n_i \cdot z(t) - b_i \leqq d\big(z(t), F_i\big) \leqq 2\,C_2\,\varepsilon \qquad \text{for all} \quad t \in [t_1, t_2], \tag{4.15}$$

and so

$$\operatorname{Osc}\big(n_i \cdot z, [t_1, t_2]\big) \leqq 2\,C_2\,\varepsilon. \tag{4.16}$$

Now, since $\mathbf{K} = \mathbf{J}$ is maximal, there is $x_0 \in \mathbf{J}$ and by **(S.b)** there exists a positive linear combination $\eta = \sum_{i \in \mathbf{J}} \gamma_i n_i$ ($\gamma_i > 0$ for all i) of the $\{n_i,\ i \in \mathbf{J}\}$ such that $\eta \cdot v_i > 0$ for all $i \in \mathbf{J}$. Then

$$\eta \cdot \big(z(t) - x_0\big) = \eta \cdot \big(x(t) - x_0\big) + \sum_{i \in \mathbf{J}} (\eta \cdot v_i) y_i(t) \qquad \text{for all} \quad t \in [0, T]. \tag{4.17}$$

Using (4.16), (4.17), and the fact that the y_i are nondecreasing, we see that one can choose a constant C_3 depending only on (S, R) such that

$$\begin{aligned} \min_{i \in \mathbf{J}} (\eta \cdot v_i) \operatorname{Osc}\big(y_1 + \cdots + y_m, [t_1, t_2]\big) &\leqq \operatorname{Osc}\big(\eta \cdot z, [t_1, t_2]\big) + \operatorname{Osc}\big(\eta \cdot x, [t_1, t_2]\big) \\ &\leqq \sum_{i \in \mathbf{J}} \gamma_i \left(\operatorname{Osc}\big(n_i \cdot z, [t_1, t_2]\big) + \operatorname{Osc}\big(n_i \cdot x, [t_1, t_2]\big)\right) \leqq C_3\, \varepsilon. \end{aligned} \tag{4.18}$$

Since $\operatorname{Osc}(y_i, [t_1, t_2]) \leqq \operatorname{Osc}(y_1 + \cdots + y_m, [t_1, t_2])$ and $z = x + Ry$, the desired oscillation estimates then follow for y and z.

(ii) There is $i \in \mathbf{J}$ and $t_3 \in [t_1, t_2]$ such that $d(z(t_3), F_i) > 2C_2\varepsilon$. Define $t_1' = \inf\{t > t_1\colon\ d(z(t), F_i) > 2C_2\varepsilon$ for some $i \in \mathbf{J}\}$. By continuity of paths, over $[t_1, t_1']$ we have the situation in part (c) (i) above. Over $[t_1', t_2]$, by Lemma 4.1, we have $z(t_1') \in F_{\mathbf{K}}^{C_1\varepsilon}$ for some $\mathbf{K} \in \mathcal{C}\backslash\{\mathbf{J}\}$, and then we have the situation in part (b). Thus, there is a constant C_4 depending only on (S, R) such that

$$\operatorname{Osc}\big(z, [t_1, t_2]\big) \leqq C_4 \varepsilon. \tag{4.19}$$

Similar reasoning to that in part (c) (i) above (cf. (4.18)) then shows that there is a constant C_5 depending only on (S, R) such that

$$\operatorname{Osc}\big(y, [t_1, t_2]\big) \leqq C_5 \varepsilon.$$

For the statement of the next theorem, we shall need the notion of a stopped SRBM with an arbitrary initial law. Such SRBM's can be obtained by time shifts and stopping of SRBM's starting from fixed points. In the following, S is endowed with the σ-field of Borel sets.

DEFINITION 4.2. Given a probability distribution μ on S, a stopped SRBM associated with (S, θ, Γ, R) having initial law μ is an $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z, together with an $\{\mathcal{F}_t\}$-stopping time T, defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that

$$Z = X + RY, \tag{4.20}$$

where

(i) P-a.s., Z has continuous paths in S and $Z(\cdot) = Z(\cdot \wedge T)$,

(ii) under P, $X(\cdot) = X(\cdot \wedge T)$ is a stopped d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t \wedge T) - \theta\,(t \wedge T),\ \mathcal{F}_{t \wedge T},\ t \geqq 0\}$ is a local martingale and $X(0)$ has distribution μ,

(iii) Y is an $\{\mathcal{F}_t\}$-adapted, m-dimensional process such that P-a.s. for each $i \in \{1, \ldots, m\}$, the ith component Y_i of Y satisfies

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and nondecreasing,

(c) $Y_i(\cdot) = Y_i(\cdot \wedge T)$,

(d) Y_i can increase only when Z is on F_i, i.e., $\int_0^t 1_{F_i}(Z(s))\, dY_i(s) = Y_i(t)$ for all $t \geqq 0$.

We say that such an SRBM is stopped at T.

Remark. The use of "local martingale" instead of "martingale" in (ii) above is necessary because of the arbitrariness of the initial distribution for X. By the martingale characterization of Brownian motion, property (ii) above can be replaced by the following equivalent condition:

(ii$'$) under P, $X(\cdot) = X(\cdot \wedge T)$, $\{X(t \wedge T) - \theta\,(t \wedge T), \mathcal{F}_{t\wedge T},\ t \geqq 0\}$ is a local martingale and for all $i, j \in \{1, \dots, d\}$, $\{(X_i(t \wedge T) - \theta_i\,(t \wedge T))\,(X_j(t \wedge T) - \theta_j\,(t \wedge T)) - \Gamma_{ij}(t \wedge T), \mathcal{F}_{t\wedge T},\ t \geqq 0\}$ is a local martingale, and $X(0)$ has distribution μ.

The following theorem was first proved by Reiman and Williams [34] when the state space S is the non-negative orthant in $\mathbf{R}^d$ and $T = \infty$.

THEOREM 4.4. *Let Z on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a stopped SRBM associated with (S, θ, Γ, R). Let T denote the stopping time at which Z is stopped. Then for any $\mathbf{K} \subset \mathbf{J}$ with $|\mathbf{K}| \geqq 2$,*

$$\int_0^T 1_{F_{\mathbf{K}}}\big(Z(s)\big)\,dY_i(s) = 0 \quad P\text{-a.s.} \quad \textit{for all} \quad i \in \mathbf{J}. \tag{4.21}$$

Remark 1. In the above theorem, the initial law of Z is not of importance and so we have not given it a name.

Remark 2. By a Girsanov transformation, it is enough to prove this theorem with $\theta = 0$ (cf. Lemma 6 of [34]). For the proof in this case, we need the following lemma. The functions used in the proof of this lemma are adapted from [34].

LEMMA 4.5. *Assume the hypotheses of Theorem 4.4 hold and in addition suppose that $d \geqq 2$, $\theta = 0$, $|\mathbf{J}| \geqq 2$ and $F_{\mathbf{J}} \neq \varnothing$. Then,*

$$\int_0^T 1_{F_{\mathbf{J}}}\big(Z(s)\big)\,dY_i(s) = 0, \quad P\text{-a.s.} \quad \textit{for all} \ \ i \in \mathbf{J}. \tag{4.22}$$

Proof. Let $\mathbf{L} \subset \mathbf{J}$ such that $\{n_i, i \in \mathbf{L}\}$ is a basis for the space spanned by $\{n_i, i \in \mathbf{J}\}$. Without loss of generality, we may suppose that $\mathbf{L} = \{1, \dots, \ell \equiv |\mathbf{L}|\}$. By the assumption **(S.b)** for $\mathbf{K} = \mathbf{J}$, there exists a positive linear combination $\eta = \sum_{i\in\mathbf{J}} \gamma_i n_i$ ($\gamma_i > 0$ for all $i \in \mathbf{J}$) of the normals $\{n_i,\ i \in \mathbf{J}\}$ such that $\eta \cdot v_i > 0$ for all $i \in \mathbf{J}$. Since $\{n_i,\ i \in \mathbf{L}\}$ is a basis, there is $\lambda = (\lambda_1, \dots, \lambda_\ell)'$ such that $\eta = \sum_{i=1}^{\ell} \lambda_i n_i$. Define $\Sigma = A\Gamma A'$, where A is the $\ell \times d$ matrix whose ith row is n_i', $i = i, \dots, \ell$. Since A has linearly independent rows, Σ is an $\ell \times \ell$ positive definite matrix. Note that $\eta = A'\lambda$ and $F_{\mathbf{L}} = F_{\mathbf{J}}$. To see the latter, note that by the basis property of $\mathbf{L}$, there are real constants a_{ij} such that $n_i = \sum_{j\in\mathbf{L}} a_{ij} n_j$ for all $i \in \mathbf{J}$, and since $F_{\mathbf{J}} \neq \varnothing$, there is $x_0 \in F_{\mathbf{J}}$ and so $n_i \cdot x_0 = b_i$ for all $i \in \mathbf{J}$. Then, if $x \in F_{\mathbf{L}}$, we have for each $i \in \mathbf{J}$,

$$n_i \cdot x = \sum_{j\in\mathbf{L}} a_{ij}\, n_j \cdot x = \sum_{j\in\mathbf{L}} a_{ij}\, b_j = \sum_{j\in\mathbf{L}} a_{ij}\, n_j \cdot x_0 = n_i \cdot x_0 = b_i,$$

and so $x \in F_{\mathbf{J}}$.

Let $\alpha = \Sigma\lambda$ and $x_0 \in F_{\mathbf{L}} = F_{\mathbf{J}}$. Then for any $x \in S$,

$$\eta \cdot (x - x_0) = \sum_{i\in\mathbf{J}} \gamma_i\, n_i \cdot (x - x_0) = \sum_{i\in\mathbf{J}} \gamma_i\, (n_i \cdot x - b_i) \geqq 0\,. \tag{4.23}$$

For each $x \in S$ and $r \in (0, 1)$, let

$$\begin{aligned} \rho^2(x, r) &= \big(A(x - x_0) + r\alpha\big)'\Sigma^{-1}\big(A(x - x_0) + r\alpha\big) \\ &= (x - x_0)'A'\Sigma^{-1}A(x - x_0) + 2r\alpha'\Sigma^{-1}A(x - x_0) + r^2\alpha'\Sigma^{-1}\alpha \end{aligned}$$

$$= (x-x_0)'A'\Sigma^{-1}A(x-x_0) + 2r\lambda' A(x-x_0) + r^2\alpha'\Sigma^{-1}\alpha$$

$$= (x-x_0)'A'\Sigma^{-1}A(x-x_0) + 2r\eta'(x-x_0) + r^2\alpha'\Sigma^{-1}\alpha \geqq r^2\widehat{\alpha}, \tag{4.24}$$

where $\widehat{\alpha} \equiv \alpha'\Sigma^{-1}\alpha = \lambda'\Sigma\lambda > 0$. For each $\varepsilon \in (0,1)$, let

$$\varphi_\varepsilon(x) \equiv \begin{cases} \dfrac{1}{2-d}\displaystyle\int_\varepsilon^1 r^{d-2}(\rho^2(x,r))^{1-d/2}\,dr, & \text{if} \quad d \geqq 3, \\ \dfrac{1}{2}\displaystyle\int_\varepsilon^1 \log(\rho^2(x,r))\,dr, & \text{if} \quad d = 2. \end{cases} \tag{4.25}$$

Observe that $\varphi_\varepsilon \in C^2(S)$. A calculation using the facts that $\theta = 0$ and

$$\operatorname{trace}\left(\Gamma A'\Sigma^{-1}A\right) = \sum_{i,j,k,l} \Gamma_{ij}\, A_{kj}\, \Sigma^{-1}_{kl}\, A_{li} = \sum_{k,l}\left(\sum_{i,j} A_{li}\,\Gamma_{ij}\,A_{kj}\right)\Sigma^{-1}_{kl}$$
$$= \sum_{k,l} \Sigma_{lk}\,\Sigma^{-1}_{kl} = d,$$

reveals that the integrand in (4.25) is L-harmonic as a function of x in some domain containing S. Hence it is readily verified that for each $\varepsilon \in (0,1)$, $L\varphi_\varepsilon = 0$ in some domain containing S. Since

$$\nabla\varphi_\varepsilon(x) = \int_\varepsilon^1 r^{d-2}A'\Sigma^{-1}\big(A(x-x_0)+r\alpha\big)\left(\rho^2(x,r)\right)^{-d/2}dr$$
$$= \int_\varepsilon^1 r^{d-2}\big(A'\Sigma^{-1}A(x-x_0)+r\eta\big)\left(\rho^2(x,r)\right)^{-d/2}dr\,,$$

then for any $i \in \mathbf{J}$,

$$D_i\varphi_\varepsilon(x) = \int_\varepsilon^1 r^{d-2}\big(v_i\cdot A'\Sigma^{-1}A(x-x_0) + rv_i\cdot\eta\big)\left(\rho^2(x,r)\right)^{-d/2}dr. \tag{4.26}$$

Let $u_i = \Sigma^{-1}Av_i$. Then $u_i \neq 0$ because $u_i\cdot\alpha = v_i\cdot\eta > 0$. Set $\delta_i = v_i\cdot\eta$ and $\beta_i = \delta_i/|u_i|$. Then similar to the derivations of (17)–(18) of [34], we have

$$D_i\varphi_\varepsilon(x) \geqq -c_i\,(\log\varepsilon + 1) \tag{4.27}$$

for all $x \in S$ satisfying $|A(x-x_0)| < \varepsilon\beta_i$, where

$$c_i = \left(\|\Sigma^{-1}\|(\beta_i + |\alpha|)^2\right)^{-d/2}\delta_i$$

and $\|\Sigma^{-1}\|$ denotes the norm of Σ^{-1} as an operator from $\mathbf{R}^\ell$ to $\mathbf{R}^\ell$. Also analogous to the derivations of (19)–(24) of [34], with $\rho_k(x) = -u_k\cdot A(x-x_0)/\delta_k$ there, we can show that there is $\widehat{c}_i \geqq 0$ such that for all $x \in S$ and $\varepsilon \in (0,1)$,

$$D_i\varphi_\varepsilon(x) \geqq -\widehat{c}_i\,. \tag{4.28}$$

For each positive integer k, define

$$\tau_k = \inf\left\{t \geqq 0\colon \left|A\big(Z(t)-x_0\big)\right| \geqq k \ \text{ or } \ Y_i(t) \geqq k \ \text{ for some } \ i \in \{1,\ldots,m\}\right\} \wedge T.$$

Observe that φ_ε and $\nabla\varphi_\varepsilon$ are bounded on $\{x \in S\colon |A(x-x_0)| \leqq k\}$ and hence in a similar manner to the derivation of (26) in [34], on applying Itô's formula to $\varphi_\varepsilon(Z(\cdot \wedge \tau_k))$ and taking expectations E with respect to P, we have

$$\mathbf{E}\left[\varphi_\varepsilon(Z(t\wedge\tau_k)) - \varphi_\varepsilon(Z(0))\right] = \sum_{i=1}^m \mathbf{E}\left[\int_0^{t\wedge\tau_k} D_i\varphi_\varepsilon(Z(s))\,dY_i(s)\right]$$

$$\geqq -(\log\varepsilon + 1)\sum_{i=1}^m c_i\,\mathbf{E}\left[\int_0^{t\wedge\tau_k} 1_{\{|A(Z(s)-x_0)|<\varepsilon\beta_i\}}\,dY_i(s)\right]$$

$$-\sum_{i=1}^m \widehat{c}_i\,\mathbf{E}\left[Y_i(t\wedge\tau_k)\right], \tag{4.29}$$

where the lower bounds (4.27) and (4.28) have been used to obtain the last inequality. Now the left side of (4.29) is uniformly bounded as $\varepsilon \to 0$ since φ_ε is uniformly bounded on $\{x \in S\colon |A(x-x_0)| \leqq k\}$ (cf. (4.24)). The last sum in (4.29) is finite by the definition of τ_k and furthermore this sum is independent of ε. Thus dividing (4.29) by $-(\log\varepsilon+1)$ and letting $\varepsilon \to 0$ yields

$$\limsup_{\varepsilon\to 0}\sum_{i=1}^m c_i\,\mathbf{E}\left[\int_0^{t\wedge\tau_k} 1_{\{|A(Z(s)-x_0)|<\varepsilon\beta_i\}}\,dY_i(s)\right] \leqq 0\,.$$

Since $F_{\mathbf{L}} \subset \{x \in S\colon A(x-x_0) = 0\}$, it follows that for each $i \in \mathbf{J}$,

$$\int_0^{t\wedge\tau_k} 1_{F_{\mathbf{L}}}(Z(s))\,dY_i(s) = 0, \quad P\text{-a.s.}$$

Letting $k \to \infty$ and $t \to \infty$, and noting that $t\wedge\tau_k \uparrow T$ in this limit, and recalling that $F_{\mathbf{L}} = F_{\mathbf{J}}$, we obtain the desired result.

Proof of Theorem 4.4. As per a remark following the statement of this theorem, we assume $\theta = 0$. Note first that since S is nonempty and has minimal descriptors, when $d = 1$, $F_{\mathbf{K}} = \varnothing$ for $|\mathbf{K}| \geqq 2$ and so the theorem holds for $d = 1$. Thus, we henceforth assume that $d \geqq 2$. Next note that it suffices to prove the theorem for $\mathbf{K}$ maximal. We shall do this by a backwards induction on $|\mathbf{K}|$. For this, let $m = |\mathbf{J}| \geqq 2$.

First observe that if $\mathbf{K} = \mathbf{J}$, then (4.21) holds by Lemma 4.5. For the backwards induction step, suppose that $2 \leqq j < m$ and (4.21) holds for all maximal $\mathbf{K}$ with $j < |\mathbf{K}| \leqq m$. Now let $\mathbf{K}$ be a maximal subset of $\mathbf{J}$ such that $|\mathbf{K}| = j$. For $\ell \in \mathbf{J}\backslash\mathbf{K}$, let $\mathbf{K}^\ell = \mathbf{K}\cup\{\ell\}$ and observe that either $F_{\mathbf{K}^\ell} = \varnothing$ or there is a maximal set $\mathbf{L} \supset \mathbf{K}^\ell$ such that $F_{\mathbf{K}^\ell} = F_{\mathbf{L}}$. In either case, using the backwards induction assumption, we have that (4.21) holds with $\mathbf{K}^\ell$ in place of $\mathbf{K}$. Combining these results for all $\ell \in \mathbf{J}\backslash\mathbf{K}$, we see that P-a.s. for each $i \in \mathbf{J}$,

$$\int_0^T 1_{F_{\mathbf{K}}}(Z(s))\,dY_i(s) = \int_0^T 1_{F_{\mathbf{K}}}(Z(s))\,1_{\{n_\ell\cdot Z(s)-b_\ell>0 \text{ for all } \ell\in\mathbf{J}\backslash\mathbf{K}\}}\,dY_i(s).$$

Then, by monotone convergence and the fact that Y_ℓ can increase only when Z is on $F_\ell = \{x \in S\colon n_\ell\cdot x = b_\ell\}$, we see that it suffices to prove for each $\varepsilon > 0$ and $i \in \mathbf{K}$,

$$\int_0^T 1_{F_{\mathbf{K}}}(Z(s))\,1_{\{n_\ell\cdot Z(s)-b_\ell\geqq\varepsilon \text{ for all } \ell\in\mathbf{J}\backslash\mathbf{K}\}}\,dY_i(s) = 0 \qquad P\text{-a.s.} \tag{4.30}$$

To verify (4.30), fix an $\varepsilon > 0$ and define sequences of stopping times $\{\sigma_k\}_{k=0}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ as follows. Let $\sigma_0 = 0$ and for $k \geqq 1$ define

$$\tau_k = \inf\left\{s \geqq \sigma_{k-1} \colon n_\ell \cdot Z(s) - b_\ell \leqq \varepsilon/2 \text{ for some } \ell \in \mathbf{J}\backslash\mathbf{K}\right\} \wedge T,$$
$$\sigma_k = \inf\left\{s \geqq \tau_k \colon n_\ell \cdot Z(s) - b_\ell \geqq \varepsilon \text{ for all } \ell \in \mathbf{J}\backslash\mathbf{K}\right\} \wedge T.$$

Now for all $i \in \mathbf{K}$,

$$\begin{aligned} &\int_0^T 1_{F_\mathbf{K}}\big(Z(s)\big)\, 1_{\{n_\ell \cdot Z(s) - b_\ell \geqq \varepsilon \text{ for all } \ell \in \mathbf{J}\backslash\mathbf{K}\}} dY_i(s) \\ &\qquad \leqq \sum_{k=0}^{\infty} \int_{\sigma_k}^{\tau_{k+1}} 1_{F_\mathbf{K}}\big(Z(s)\big)\, dY_i(s). \end{aligned} \tag{4.31}$$

Now P-a.s., for each $\ell \in \mathbf{J}\backslash\mathbf{K}$, Y_ℓ does not increase on $[\sigma_k, \tau_{k+1}]$ and so on $\{\sigma_k < \infty\}$ we have P-a.s.,

$$\begin{aligned} Z\big((\cdot + \sigma_k) \wedge \tau_{k+1}\big) &= Z(\sigma_k) + X\big((\cdot + \sigma_k) \wedge \tau_{k+1}\big) - X(\sigma_k) \\ &\quad + \sum_{i \in \mathbf{K}} v_i \Big(Y_i\big((\cdot + \sigma_k) \wedge \tau_{k+1}\big) - Y_i(\sigma_k)\Big). \end{aligned}$$

It follows that $Z((\cdot + \sigma_k) \wedge \tau_{k+1}) 1_{\{\sigma_k < \infty\}}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_{t+\sigma_k}\}, P)$ is a stopped SRBM associated with $(S^\mathbf{K}, \theta, \Gamma, R^\mathbf{K})$ where $S^\mathbf{K}$, $R^\mathbf{K}$ are as defined before Lemma 4.2. The stopping time for this stopped SRBM is $\tau \equiv (\tau_{k+1} - \sigma_k) 1_{\{\sigma_k < \infty\}}$. Since $\mathbf{K}$ is maximal, by Lemma 4.2, we may apply Lemma 4.5 with $\mathbf{K}$ in place of $\mathbf{J}$ to conclude that P-a.s. on $\{\sigma_k < \infty\}$,

$$\int_{\sigma_k}^{\tau_{k+1}} 1_{F_\mathbf{K}}\big(Z(s)\big)\, dY_i(s) = \int_0^\tau 1_{F_\mathbf{K}}\big(Z(s + \sigma_k)\big)\, dY_i(s + \sigma_k) = 0 \qquad \text{for all} \quad i \in \mathbf{K}.$$

Thus, the right member of (4.31) is zero P-a.s. for all $i \in \mathbf{K}$ and so (4.30) holds. This completes the induction step and so the theorem is proved.

The following lemma is a generalization of Lemma 2.1 of [37], where S was the non-negative orthant in $\mathbf{R}^d$. Here S^0 denotes the interior of S and F_i° denotes the relative interior of F_i, i.e., $F_i^\circ = F_i \backslash \cup_{j \neq i} F_j$.

LEMMA 4.6. *Let Z, T, P be as described in Theorem 4.4. Then P-a.s.,*

$$\int_0^T 1_{\partial S}\big(Z(s)\big)\, ds = 0, \tag{4.32}$$

$$X(t \wedge T) = Z(0) + \int_0^{t \wedge T} 1_{S^0}\big(Z(s)\big)\, dZ(s) \qquad \textit{for all} \quad t \geqq 0, \tag{4.33}$$

$$\begin{aligned} Y_i(t \wedge T) &= (n_i \cdot v_i)^{-1} \int_0^{t \wedge T} 1_{F_i^\circ}\big(Z(s)\big)\, d(n_i \cdot Z)(s) \\ &\qquad \textit{for all} \quad t \geqq 0,\ i \in \mathbf{J}. \end{aligned} \tag{4.34}$$

Proof. For (4.32), it suffices to prove that for each $i \in \{1, 2, \ldots, m\}$, P-a.s.,

$$\int_0^T 1_{\{b_i\}}\big(n_i \cdot Z(s)\big)\, ds = 0, \qquad P\text{-a. s.} \tag{4.35}$$

For this, one can proceed as in the proof of Lemma 2.1 of [37]. In particular, under P, $n_i \cdot Z$ is a continuous one-dimensional semimartingale and so

$$\int_0^T 1_{\{b_i\}}\big(n_i \cdot Z(s)\big)\, d[n_i \cdot Z]_s = \int_{\mathbf{R}} L^y_T 1_{\{y=b_i\}}\, dy = 0\,, \tag{4.36}$$

where $L^y_\cdot$ is the local time at $y \in \mathbf{R}$ of the continuous semimartingale $n_i \cdot Z$. Since the quadratic variation $[n_i \cdot Z]_s = [n_i \cdot X]_s = n_i'\Gamma n_i(s \wedge T)$, where $n_i'\Gamma n_i > 0$, (4.32) follows.

The representation (4.33) follows exactly as in Lemma 2.1 of [37], using the facts that dX does not charge the set of times of zero Lebesgue measure for which Z is on ∂S and that none of the Y_i can increase when Z is in the interior of S.

The proof of (4.34) is very similar to that in Lemma 2.1 of [37], except that one uses Theorem 4.4 in place of [34] to conclude that P-a.s. for all $t \geqq 0$ and $i \in \mathbf{J}$,

$$Y_i(t \wedge T) = \int_0^{t\wedge T} 1_{F_i^\circ}\big(Z(s)\big)\, dY_i(s). \tag{4.37}$$

Then one uses the fact that (**S.a**) holds for $\mathbf{K} = \{i\}$ to conclude that $n_i \cdot v_i > 0$, and thus one can replace dY_i by $(n_i \cdot v_i)^{-1} d(n_i \cdot Z)$ in the above to obtain (4.34).

5. Uniqueness. Our proof of the uniqueness claimed in Theorem 1.3 follows the same general lines as in [37] where the case $S = \mathbf{R}^d_+$ was treated. Accordingly, we shall indicate the general outline of the argument and only go into the details of the proofs when they differ from those in [37].

5.1. Preliminaries and the induction hypothesis. The following tightness result will be used in the proof of uniqueness and Feller continuity in Theorem 1.3.

LEMMA 5.1. *For each $x \in S$, let Q_x denote a probability measure induced on $(\mathbf{C}, \mathcal{M})$ by an SRBM and its associated pushing process for the data (S, θ, Γ, R) and starting point x (cf. (1.4)). Fix $x_0 \in S$ and let $\{x_n\}_{n=1}^\infty$ be a sequence in S such that $x_n \to x_0$ as $n \to \infty$. Then the sequence $\{Q_{x_n},\ n = 1, 2, \ldots\}$ of probability measures on $(\mathbf{C}, \mathcal{M})$ is tight. Any weak limit point P_{x_0} of this sequence together with the canonical process $z(\cdot)$ on $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\})$ defines an SRBM associated with (S, θ, Γ, R) that starts from x_0, where the attendant pushing process is given by the other canonical process $y(\cdot)$.*

Proof. This lemma can be proved in a similar manner to Theorem 5.4 of [37]. In particular, the tightness follows from the oscillation Lemma 4.3 together with the tightness for Brownian motions with starting points lying in a compact set. For the SRBM property of (z, y) under P_{x_0}, note that properties (1.3), (i), (ii) (for $X = z - Ry$), (iii) (a), and (iii) (b), of Definition 1.1 follow from the weak convergence. For property (iii) (c), since P_{x_0}-a.s., $n_i \cdot z(s) - b_i \geqq 0$ for all $s \geqq 0$, where equality holds only if $z(s) \in F_i$, and y_i is nondecreasing, it suffices to prove that for each $i \in \mathbf{J}$,

$$\int_0 \big((n_i \cdot z(s) - b_i) \wedge 1\big)\, dy_i(s) = 0\,, \qquad P_{x_0}\text{-a. s.} \tag{5.1}$$

To see this, fix $t \geqq 0$. It follows from Lemma 2.4 that

$$\int_0^t \big((n_i \cdot z(s) - b_i) \wedge 1\big)\, dy_i(s) \tag{5.2}$$

is a continuous bounded function defined on the set of $(z, y) \in \mathbf{C}$, where y is restricted to the set of nondecreasing functions. Now for each n, (5.2) is zero Q_{x_n}-a.s. It then follows from the weak convergence and the continuous mapping theorem that the same applies under P_{x_0}.

THEOREM 5.2. *Theorem* 1.3 *holds for* $d = 1$.

Proof. The existence has already been established. For the uniqueness, let Z under P_x be an SRBM associated with (S, θ, Γ, R) that starts from $x \in S$. Let Y denote the pushing process associated with Z. For $d = 1$, S is a half-line or a bounded interval. In either case there are unique continuous path-to-path mappings φ, ψ from $\{x \in C_{\mathbf{R}}[0, \infty)\colon x(0) \in \mathbf{R}_+\}$ into $C_{\mathbf{R}_+}[0, \infty)$ such that P_x-a.s., $Z = \varphi(X)$ and $Y = \psi(X)$, cf. [19], [8]. Since the distribution of X under P_x is known to be that of a (θ, Γ)-Brownian motion starting from x, it follows that the law of $(Z, Y) = (\varphi(X), \psi(X))$ under P_x is uniquely determined. The Feller continuity of the associated measures $\{Q_x,\ x \in S\}$ on $(\mathbf{C}, \mathcal{M})$ follows from the continuity of the mappings φ, ψ and the Feller continuity for Brownian motion. The strong Markov property follows from the pathwise uniqueness together with the strong Markov property of Brownian motion.

Next we fix $d \geqq 2$ and make the following induction hypothesis.

Induction hypothesis. Theorem 1.3 holds for all state spaces S in $\mathbf{R}^k$ with $k \leqq d-1$.

We shall prove that Theorem 1.3 holds for state spaces S in $\mathbf{R}^d$. Note that we have already established existence. Also, by a Girsanov transformation argument, which is precisely the same as that given in Lemma 6.1 of [37], it suffices to prove the uniqueness stated in Theorem 1.3 for $\theta = 0$. Furthermore, observe that for Z, Y, P_x as described in Theorem 1.3, by Lemma 4.6, the pushing process Y is P_x-a.s. a functional of Z. Thus for the uniqueness, it suffices to prove that for each $x \in S$, an SRBM associated with $(S, 0, \Gamma, R)$ that starts from x is unique in law.

5.2. Uniqueness in a cone. We first treat the situation where S is a cone, i.e., where $F_{\mathbf{J}} \neq \varnothing$ and so

$$S = \left\{x \in \mathbf{R}^d\colon\ n_i \cdot (x - x_0) \geqq 0 \ \text{ for all } \ i \in \mathbf{J}\right\}, \tag{5.3}$$

where $x_0 \in F_{\mathbf{J}}$. Indeed, without loss of generality, we may and do assume that $x_0 = 0$ and consequently $b_i = 0$ for all $i \in \mathbf{J}$.

THEOREM 5.3. *Suppose* $F_{\mathbf{J}} \neq \varnothing$ *and the dimension of the vector space spanned by* $\{n_i,\ i \in \mathbf{J}\}$ *is less than d. Then Theorem* 1.3 *holds.*

Proof. Existence was established previously. For the uniqueness, we assume that $\theta = 0$. Now, fix $x \in S$ and let Z under P_x be an SRBM associated with $(S, 0, \Gamma, R)$ that starts from x. Let Y be the pushing process associated with Z.

Let $\mathbf{L} \subset \mathbf{J}$ such that $\{n_i,\ i \in \mathbf{L}\}$ is a basis for $\{n_i,\ i \in \mathbf{J}\}$. Without loss of generality we assume that $\mathbf{L} = \{1, \ldots, \ell \equiv |\mathbf{L}|\}$. Furthermore, since $\{n_1, \ldots, n_\ell\}$ span an ℓ-dimensional subspace of $\mathbf{R}^d$, by performing a change of basis we may assume that this subspace is the one generated by the first ℓ orthonormal basis vectors $e_1, \ldots, e_\ell$ in $\mathbf{R}^d$, and so for each $i \in \mathbf{J}$, $n_i = (\widehat{n}_i, 0)$, where $\widehat{n}_i$ denotes the projection of n_i onto $\mathbf{R}^\ell$ and 0 denotes the zero vector in $\mathbf{R}^{d-\ell}$. (Note that here we do not assume $n_1, \ldots, n_\ell$ are $e_1, \ldots, e_\ell$, since the former may be nonorthogonal.) Then,

$$S = \left\{x = (\widehat{x}, \widetilde{x})' \in \mathbf{R}^\ell \times \mathbf{R}^{d-\ell}\colon\ \widehat{n}_i \cdot \widehat{x} \geqq 0 \ \ \forall i \in \mathbf{J}\right\} \equiv \widehat{S} \times \mathbf{R}^{d-\ell}. \tag{5.4}$$

We partition R, Γ, Z, X, x accordingly, e.g., $\widehat{R}$ denotes the $\ell \times m$ matrix whose m columns are obtained from those of R by keeping the first ℓ coordinates of each column, $\widehat{\Gamma}$ is the $\ell \times \ell$ matrix in the upper left corner of the matrix Γ. Thus,

$$R = \begin{pmatrix} \widehat{R} \\ \widetilde{R} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \widehat{\Gamma} & \Delta \\ \Delta' & \widetilde{\Gamma} \end{pmatrix}, \tag{5.5}$$

$Z = (\widehat{Z}, \widetilde{Z})'$, $X = (\widehat{X}, \widetilde{X})'$, and $x = (\widehat{x}, \widetilde{x})'$. Then it can be readily verified that $(\widehat{S}, \widehat{R})$ satisfy conditions **(S.a)** and **(S.b)**, and

$$\widehat{Z} = \widehat{X} + \widehat{R}Y$$

is an $(\widehat{S}, 0, \widehat{\Gamma}, \widehat{R})$-SRBM starting from $\widehat{x}$ under P_x. By the induction assumption, we have uniqueness in law of $(\widehat{Z}, Y)$ under P_x.

One can then establish that the law of (Z, Y) under P_x is unique, where $Z = (\widehat{Z}, \widetilde{Z})'$, using the same approach as in Theorem 3.4 of [37]. In particular, the argument given in [37] for $|\mathbf{K}| = d - 1$ easily generalizes to any $|\mathbf{K}|$ with $|\mathbf{K}| < d$. Briefly, what one does is to approximate $\widetilde{Z}$ by $\widetilde{Z}^n = \widetilde{X}^n + \widetilde{R}Y$ where $\widetilde{X}^n$ approximates the Brownian motion $\widetilde{X}$ in such a way that $\widetilde{X}^n$ and Y change on disjoint stochastic intervals. When this is done in a suitable way, it can be readily shown using the uniqueness in law of $(\widehat{Z}, Y)$ and the known properties of $\widetilde{X}^n$, that the joint law of $(\widehat{Z}, Y, \widetilde{X}^n)$ under P_x is uniquely determined. Passage to the limit then gives the uniqueness in law for $(\widehat{Z}, Y, \widetilde{X})$ under P_x and hence of $(Z, Y) = (\widehat{Z}, \widetilde{X} + \widetilde{R}Y, Y)$. For more details we refer the reader to [37].

Finally, the Feller continuity and strong Markov property follow by standard arguments (cf. [36, Cor. 4.6]), using the uniqueness in law just established and the tightness of the laws given by Lemma 5.1.

Remark. Recall the notion of a stopped SRBM from Definition 4.2. Under the conditions of the last theorem, by the Feller continuity of the laws induced on the space $(\mathbf{C}, \mathcal{M})$, one can always extend the joint law of a stopped SRBM and its attendant pushing process to that of an unstopped SRBM and its attendant pushing process (cf. [37, § 4.2] and [36, Theorem 6.1.2]). It follows that under the conditions of the above theorem, if Z is a stopped SRBM starting from $x \in S$, where the stopping time T is a stopping time relative to the filtration generated by Z, and Y is the pushing process associated with Z, then the law of $(Z(\cdot \wedge T), Y(\cdot \wedge T))$ is unique.

Assumption. For the remainder of this section up to Theorem 5.14, we assume that $F_{\mathbf{J}} = \{0\}$ and $\theta = 0$.

DEFINITION 5.1. An SRBM associated with $(S, 0, \Gamma, R)$ that starts from x and is absorbed at the origin is a stopped SRBM associated with this data that starts from x and for which the stopping time $T = \inf\{t \geqq 0\colon\ Z(t) = 0\}$.

THEOREM 5.4. *Fix $x \in S$ and let Z together with P_x define an SRBM associated with $(S, 0, \Gamma, R)$ that starts from x and is absorbed at the origin. Let Y denote the associated pushing process. Let $\tau_0 = \inf\{t \geqq 0\colon\ Z(t) = 0\}$. Then the probability measure Q_x° induced on $(\mathbf{C}, \mathcal{M})$ by $(Z(\cdot \wedge \tau_0), Y(\cdot \wedge \tau_0))$ under P_x is unique, i.e., the law of an SRBM and its attendant pushing process for the data $(S, 0, \Gamma, R)$ and starting point x with absorption at the origin is unique.*

Proof. The case $x = 0$ is trivial, so we assume $x \neq 0$. Without loss of generality we suppose that (Z, Y) is the canonical pair of processes on $(\mathbf{C}, \mathcal{M})$ and $P_x = Q_x^\circ$. This ensures that we are on a sufficiently nice space to be able to take regular conditional probability distributions.

Recall the definition of C_ε from § 4. Now for each $\varepsilon > 0$, define

$$\tau_\varepsilon = \inf\left\{t \geqq 0\colon\ n_i \cdot Z(t) \leqq C_\varepsilon \ \text{ for all } \ i \in \mathbf{J}\right\}. \tag{5.6}$$

By the continuity of the paths of (Z, Y) and the definitions of τ_0 and C_ε, it suffices to prove that for each $\varepsilon > 0$, the law of $(Z(\cdot \wedge \tau_\varepsilon), Y(\cdot \wedge \tau_\varepsilon))$ under P_x is unique. Thus, we fix $\varepsilon > 0$ and without loss of generality assume that ε is sufficiently small that

$n_i \cdot x > C_\varepsilon$ for at least one $i \in \mathbf{J}$. Now by Lemma 4.1,

$$S \backslash \{w \in S\colon\ n_i \cdot w \leqq C_\varepsilon \text{ for all } i \in \mathbf{J}\} \subset \bigcup_{\mathbf{K} \in \mathcal{J}} F^\varepsilon_{\mathbf{K}}, \tag{5.7}$$

where $\mathcal{J}$ denotes the collection consisting of the empty set, together with all maximal subsets of $\mathbf{J}$, except $\mathbf{J}$. Now for each $\mathbf{K} \in \mathcal{J}$, define

$$\begin{aligned} \widetilde{F}^\varepsilon_{\mathbf{K}} = \{w \in \mathbf{R}^d\colon\ 0 \leqq n_i \cdot w \leqq 2C_\varepsilon \quad &\text{for all } i \in \mathbf{K} \text{ and } n_i \cdot w > \varepsilon/2 \\ &\text{for all } i \in \mathbf{J}\backslash\mathbf{K}\}. \end{aligned} \tag{5.8}$$

Order the sets in $\mathcal{J}$. Then let $\sigma_0 = 0$ and define a sequence of pairs $\{(\mathbf{r}_n, \sigma_n)\}_{n=1}^\infty$ by induction as follows. Let $\mathbf{r}_1$ be the first $\mathbf{K} \in \mathcal{J}$ such that $x \in F^\varepsilon_{\mathbf{K}}$, and let $\sigma_1 = \inf\{t \geqq 0\colon Z(t) \notin \widetilde{F}^\varepsilon_{\mathbf{r}_1}\}$. Assuming $\mathbf{r}_n, \sigma_n$ have been defined for some $n \geqq 1$, on $\{\sigma_n < \infty, Z(\sigma_n) \notin F^\varepsilon_{\mathbf{J}}\}$, let $\mathbf{r}_{n+1}$ be the first $\mathbf{K} \in \mathcal{J}$ such that $Z(\sigma_n) \in F^\varepsilon_{\mathbf{K}}$ and let $\sigma_{n+1} = \inf\{t \geqq \sigma_n\colon Z(t) \notin \widetilde{F}^\varepsilon_{\mathbf{r}_{n+1}}\}$. On $\{\sigma_n = \infty \text{ or } Z(\sigma_n) \in F^\varepsilon_{\mathbf{J}}\}$, define $\mathbf{r}_{n+1} = \mathbf{r}_n$ and $\sigma_{n+1} = \sigma_n$. By the continuity of the paths of Z, P_x-a.s., $\sigma_n \to \tau_\varepsilon$ as $n \to \infty$. Thus, it suffices to prove the uniqueness in law of $(Z(\cdot \wedge \sigma_n), Y(\cdot \wedge \sigma_n))$ under P_x for each n. For this, recall the definition of $(S^{\mathbf{K}}, R^{\mathbf{K}})$ from just before Lemma 4.2. For $n \geqq 0$, on $\{\sigma_n < \infty\}$, $(Z((\cdot + \sigma_n) \wedge \sigma_{n+1}), Y((\cdot + \sigma_n) \wedge \sigma_{n+1}))$ under the conditional law $P_x(\cdot\,|\mathcal{M}_{\sigma_n})$ is a stopped SRBM associated with $(S^{\mathbf{r}_{n+1}}, 0, \Gamma, R^{\mathbf{r}_{n+1}})$ that starts from $Z(\sigma_n)$. When $\mathbf{r}_{n+1} = \varnothing$, we interpret an SRBM associated with $(S^{\mathbf{r}_{n+1}}, 0, \Gamma, R^{\mathbf{r}_{n+1}})$ to be a $(0, \Gamma)$ Brownian motion and so in this case the conditional law mentioned above is simply the law of a stopped Brownian motion starting from $Z(\sigma_n)$, which is unique. In all other cases, $\mathbf{r}_{n+1}$ is a maximal subset of $\mathbf{J}$ that is not equal to $\mathbf{J}$ and in particular the dimension of the space spanned by $\{n_i,\ i \in \mathbf{r}_{n+1}\}$ must be less than d (otherwise, $F_{\mathbf{r}_{n+1}}$ would be the origin and $\mathbf{r}_{n+1} \neq \mathbf{J}$ could not be maximal). It then follows from Lemma 4.2 and the remark following Theorem 5.3 that the aforementioned conditional law is unique. Proceeding by induction, one can show that for each n, the law of $(Z(\cdot \wedge \sigma_n), Y(\cdot \wedge \sigma_n))$ under P_x is unique, as desired. For more details, the reader is referred to a similar proof of Theorem 4.3 in [37].

Recall that we already have existence of an SRBM associated with $(S, 0, \Gamma, R)$ starting from any point in S. By stopping such an SRBM at the first time it reaches the origin, we obtain an SRBM with absorption at the origin. The above lemma shows that the law of such an SRBM is unique. Furthermore, in a similar manner to that in § 4.2 of [37], one can alternatively construct this law by patching together the path space measures associated with SRBM's in the spaces $S^{\mathbf{K}}$ for $\mathbf{K} \in \mathcal{J}$. The following lemma can be established using this uniqueness and alternative construction, in a similar manner to that in which Lemma 4.5 and Corollary 4.6 of [37] were proved. Here and henceforth, for each $x \in S$, we let $\mathring{Q}_x$ denote the law defined in Theorem 5.4.

LEMMA 5.5. *For each $x \in S$, let E_x denote expectation under $\mathring{Q}_x$. Then, for each bounded continuous function h: $S \times \mathbf{R}^m_+ \to \mathbf{R}$ and $t \geqq 0$, $x \to E_x[h(z(t), y(t))]$ is a Borel measurable function on S. Furthermore, for each $f \in C_b(S)$, $\{\mathcal{M}_t\}$-stopping time T, and $t \geqq 0$,*

$$E_x\Big[1_{\{T<\infty\}} f\big(z(T+t)\big) \mid \mathcal{M}_T\Big] = 1_{\{T<\infty\}} E_{z(T)}\Big[f\big(z(t)\big)\Big]. \tag{5.9}$$

Hence, $z(\cdot)$ together with $\{\mathring{Q}_x,\ x \in S\}$ defines a strong Markov process.

Remark. The measurability cited first in the above lemma is used to ensure that the right member of (5.9) is $\mathcal{M}_T$-measurable.

The following lemma can be proved in a similar manner to Theorem 4.4 of [37], using Lemma 4.3 for the oscillation estimate.

LEMMA 5.6. *Let K denote a compact subset of S. Then the family $\{Q_x^\circ,\ x \in K\}$ of probability measures on $(\mathbf{C}, \mathcal{M})$ is tight.*

The following lemma can be proved in the same way as Theorem 4.7 of [37] using the uniqueness established in Theorem 5.4.

LEMMA 5.7. *For each $r > 0$ and $x \in S = \{w \in \mathbf{R}^d\colon n_i \cdot w \geqq 0$ for all $i \in \mathbf{J}\}$,*

$$Q_x^\circ(A) = Q_{rx}^\circ\big(r^{-1}(z,y)\,r^2 \in A\big) \qquad \text{for each} \quad A \in \mathcal{M}. \tag{5.10}$$

The following two technical lemmas are critical to our proof of the compactness result in Lemma 5.11 and in turn the ergodic property in Lemma 5.12. The latter is used in an essential way to prove uniqueness when $F_{\mathbf{J}} = \{0\}$.

LEMMA 5.8. *Fix $x_0 \in S \backslash \{0\}$. For each $r \geqq 0$, let $\zeta_r = \inf\{t \geqq 0\colon |z(t) - x_0| \geqq r\}$ be defined on $\mathbf{C}$. There are constants $\rho > 0$, $\kappa > 0$, $\gamma \in (0, \frac{1}{2} \wedge |x_0|/2]$ and $\beta \in (0, \frac{1}{2})$ such that for each r satisfying $0 < r \leqq \gamma$ and $x \in S$ satisfying $|x - x_0| \leqq \beta r$, we have*

$$Q_x^\circ\Big(d\big(z(\zeta_r), \partial S\big) > \rho r\Big) \geqq \kappa > 0. \tag{5.11}$$

Remark. The constants ρ, κ, γ, and β may depend on x_0, but not on r.

Proof. Let $\mathcal{J}$ be defined as in the proof of Theorem 5.4. By Lemma 4.1, since $F_{\mathbf{J}} = \{0\}$ and $x_0 \neq 0$, $x_0 \in F_{\mathbf{K}} \backslash (\cup_{i \notin \mathbf{K}} F_i)$ for some $\mathbf{K} \in \mathcal{J}$. We shall prove that the lemma holds, by induction on $|\mathbf{K}|$.

If $|\mathbf{K}| = 0$, i.e., $\mathbf{K} = \varnothing$, then $x_0 \in S^\circ$ and the result follows easily with $\rho = \frac{1}{2}$, $\gamma = \frac{1}{2} d(x_0, \partial S) \wedge \frac{1}{2}$, any $\beta \in (0, \frac{1}{2})$, and $\kappa = 1$, since then under $Q_{x_0}^\circ$, $z(\cdot)$ behaves like a $(0, \Gamma)$-Brownian motion until it reaches $\partial B(x_0, \gamma) \subset S^0$.

Now, for the induction step, assume the result holds for all $\mathbf{K} \in \mathcal{J}$ satisfying $|\mathbf{K}| \leqq k$ for some $k \in \{0, \dots, m-2\}$. Then, consider $\mathbf{K} \in \mathcal{J}$ satisfying $|\mathbf{K}| = k+1$. For $F \equiv \cup_{i \notin \mathbf{K}} F_i$, we have $d(x_0, F) > 0$. Let $(S^{\mathbf{K}}, R^{\mathbf{K}})$ be defined as in § 4 with $b = 0$. Note that from Lemma 4.2, conditions **(S.a)** and **(S.b)** hold for this pair. Furthermore, since $x_0 \neq 0$, the dimension of the vector space spanned by $\{n_i,\ i \in \mathbf{K}\}$ is less than d. Thus, we can apply Theorem 5.3 with $\mathbf{K}$ in place of $\mathbf{J}$ to conclude that Theorem 1.3 applies with the SRBM data $(S^{\mathbf{K}}, 0, \Gamma, R^{\mathbf{K}})$. The following explicit representation for the SRBM associated with this data is needed for the proof of the current lemma.

Let $\widehat{\mathbf{K}} \subset \mathbf{K}$ such that $\{n_i,\ i \in \widehat{\mathbf{K}}\}$ is a basis for the space spanned by $\{n_i,\ i \in \mathbf{K}\}$. Then $\ell \equiv |\widehat{\mathbf{K}}| < d$, since $x_0 \neq 0$. In a similar manner to that in the proof of Theorem 5.3, we may obtain a representation for $S^{\mathbf{K}}$ of the form (5.4) with $\mathbf{K}$ in place of $\mathbf{J}$ there. More precisely, we assume without loss of generality that $\widehat{\mathbf{K}} = \{1, \dots, \ell\}$ and the span of $\{n_1, \dots, n_\ell\}$ is the ℓ-dimensional subspace of $\mathbf{R}^d$ generated by the first ℓ orthonormal basis vectors in $\mathbf{R}^d$, and so

$$S^{\mathbf{K}} = \widehat{S} \times \mathbf{R}^{d-\ell}, \tag{5.12}$$

where

$$\widehat{S} = \{\widehat{x} \in \mathbf{R}^\ell\colon \widehat{n}_i \cdot \widehat{x} \geqq 0 \text{ for all } i \in \mathbf{K}\}, \tag{5.13}$$

and $\widehat{n}_i$ denotes the projection of n_i onto $\mathbf{R}^\ell$. We assume that $R^{\mathbf{K}}$ and Γ have been decomposed in a similar manner to that in (5.5). It is readily verified that $(\widehat{S}, \widehat{R})$ satisfy conditions **(S.a)** and **(S.b)**. Since $\ell < d$, by Theorem 5.3 there is an SRBM associated with $(\widehat{S}, 0, \widehat{\Gamma}, \widehat{R})$. We suppose that $\widehat{Z} = \widehat{X} + \widehat{R}\widehat{Y}$ with an associated family of probability measures $\{\widehat{P}_{\widehat{x}},\ \widehat{x} \in \widehat{S}\}$ defined on some filtered space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\})$ is such an SRBM.

In a similar manner to that in Proposition 3.2 and Theorem 3.3 of [37] with $\widehat{\Gamma}, \widetilde{\Gamma}, \Delta$ in place of $\Gamma_{\mathbf{K}}, \Gamma_{|\mathbf{K}}, \Lambda$, respectively, on some enlargement of the filtered space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\})$, from the SRBM $(\widehat{Z}, \{\widehat{P}_{\widehat{x}},\ \widehat{x} \in \widehat{S}\})$ and an independent $(d-\ell)$-dimensional Brownian motion $(\widetilde{B}, \{\widetilde{P}_{\widetilde{x}},\ \widetilde{x} \in \mathbf{R}^{d-\ell}\})$, one can construct an SRBM $(Z, \{P_x^{\mathbf{K}},\ x \in S^{\mathbf{K}}\})$ associated with $(S^{\mathbf{K}}, 0, \Gamma, R^{\mathbf{K}})$. The proof of the lemma then proceeds in almost exactly the same manner as the proof of Lemma 6.2 of [37], with the following exceptions. One uses $F = \cup_{i \notin \mathbf{K}} F_i$, $R_{\mathbf{K}} = \widehat{R}$, $\Gamma_{\mathbf{K}} = \widehat{\Gamma}$, $P_x = P_x^{\mathbf{K}}$, $\widehat{S}$ in place of $\mathbf{R}_+^k$, $\mathbf{L} \in \mathcal{J}$ in place of $\mathbf{L}_0$, Lemma 4.3 in place of the oscillation estimates of Bernard and El Kharroubi [4], Theorem 5.3 with data $(S^{\mathbf{K}}, 0, \Gamma, R^{\mathbf{K}})$ for the strong Markov property of $Z^{\mathbf{K}}(\cdot)$ under $P_x^{\mathbf{K}}$, and Lemma 5.7 for scaling in place of Theorem 4.7. In place of the vector v one uses a vector $\eta \in \mathbf{R}^\ell$ which by condition **(S.b)** can be chosen as a positive linear combination of the vectors $\widehat{n}_i$, $i \in \mathbf{K}$, such that $|\eta| = 1$ and $\eta'\widehat{R} > 0$. Note for this that $\eta \cdot (\widehat{x} - \widehat{x}_0) \geqq 0$ for any $x \in S^{\mathbf{K}}$. Finally, the constant ρ replaces the constant $\frac{1}{8}d$ which appears in the definition of $\mathcal{I}_r$ in Lemma 6.2 of [37]. This constant $\rho > 0$ needs to be chosen sufficiently small that $\{x \in S\colon\ d(x, \partial S) > \rho\gamma,\ |x - x_0| = \gamma\}$ for $\gamma = \frac{1}{2}d(x_0, F) \wedge \frac{1}{2}$ has positive surface measure as a subset of $\partial B(x_0, \gamma)$.

We note in passing that in the proof of Lemma 6.2, the balls B_u, B_{u_i} following the definition of $\widehat{\sigma}$ should have radii $\gamma(u), \gamma(u_i)$, respectively.

Let $\Lambda \equiv \{x \in S\colon\ |x| = 1\}$.

LEMMA 5.9. *Fix $x_0 \in \Lambda$ and for each $r > 0$, let ζ_r be defined as in Lemma 5.8. There are constants $\kappa > 0$, $\gamma \in (0, \frac{1}{2}]$, and $\beta \in (0, \frac{1}{4})$ such that for each r satisfying $0 < r \leqq \gamma$ and $x \in S$ satisfying $|x - x_0| \leqq \beta r$, we have*

$$Q_x^\circ\big(z(\zeta_r) \in A_r\big) \geqq \kappa, \tag{5.14}$$

whenever $A_r \subset S \cap \partial B(x_0, r)$ such that $|A_r| \geqq \frac{1}{2}|S \cap \partial B(x_0, r)|$. Here $|\cdot|$ denotes surface measure on $\partial B(x_0, r)$. The constants κ, γ, and β can be chosen to be independent of r.

Proof. This can be proved in a similar manner to Lemma 6.3 of [37] using Lemma 5.8 above in place of Lemma 6.2 of [37]. One minor adjustment is that for the proof of the current lemma, $\mathcal{I}_r$ in [37] should be defined as $\{x \in S\colon\ d(x, \partial S) > \rho r\}$ and $\frac{1}{32}d$ should be replaced by $\frac{1}{4}\rho$.

For each $x \in \Lambda$, define the subprobability measure $Q(x, \cdot)$ on the Borel σ-field $\mathcal{B}(\Lambda)$ of Λ by

$$Q(x, A) = Q_x^\circ\left(\frac{z(\tau_2)}{2} \in A, \tau_2 < \tau_0\right) \qquad \text{for all} \quad A \in \mathcal{B}(\Lambda), \tag{5.15}$$

where $\tau_r = \inf\{t \geqq 0\colon\ |z(t)| = r\}$ for $r \geqq 0$. We now record several properties of Q.

LEMMA 5.10. *For $x \in S\backslash\{0\}$ and $r = |x|$,*

$$Q_x^\circ\left(\frac{z(\tau_{2r})}{2}r \in A, \tau_{2r} < \tau_0\right) = Q\left(\frac{x}{|x|}, A\right) \qquad \textit{for all} \quad A \in \mathcal{B}(\Lambda).$$

Proof. This scaling property follows immediately from Lemma 5.7.

Let $C(\Lambda)$ denote the space of all (bounded) continuous real-valued functions defined on Λ and let $C(\Lambda)$ be endowed with the sup-norm topology. Define

$$(Qf)(x) = \int_\Lambda Q(x, dy) f(y) \qquad \text{for all} \quad x \in \Lambda \text{ and } f \in C(\Lambda). \tag{5.16}$$

LEMMA 5.11. *For each $f \in C(\Lambda)$, $Qf \in C(\Lambda)$. Moreover, Q is a compact operator on $C(\Lambda)$.*

Proof. This lemma is proved in the same manner as Theorem 3.2 of [29] except that in our proof, Lemma 5.9, β, ζ_r are used in place of Lemma 3.3, $\frac{1}{4}$, η_r, of [29].

We shall denote the nth power of the operator Q: $C(\Lambda) \to C(\Lambda)$ by Q^n. The following is an ergodic result which is key to our proof of uniqueness when $F_{\mathbf{J}} = \{0\}$.

LEMMA 5.12. *Suppose G and H are continuous, bounded, real-valued functions on Λ such that $H \geqq 0$ and $H \not\equiv 0$. Let $\{\nu_n\}$ be a sequence of probability measures on $(\Lambda, \mathcal{B}(\Lambda))$. Then*

$$\frac{\int_\Lambda (Q^n G)(x)\, \nu_n(dx)}{\int_\Lambda (Q^n H)(x)\, \nu_n(dx)} \to C(G,H) \qquad \text{as} \qquad n \to \infty, \tag{5.17}$$

where $C(G,H)$ is a finite constant depending only on Q, G, H, and not on the sequence $\{\nu_n\}$.

Proof. This can be proved in an analogous manner to Lemma 6.7 of [37]. In particular, one verifies the hypotheses of the Krein–Rutman theorem using Lemmas 5.8 and 5.11 in place of Lemmas 6.2 and 6.6 in [37], $\mathcal{I}_r = \{x \in S\colon\ d(x, \partial S) > \rho r\}$, and in the definition of U, B_{u_i} should read $B(u_i, \gamma(u_i))$.

LEMMA 5.13. *On $(\mathbf{C}, \mathcal{M})$, for each $r > 0$ define $\tau_r = \inf\{t \geqq 0\colon\ |z(t)| = r\}$. There is a finite constant C such that for each $r > 0$, any $x \in S$ satisfying $|x| \leqq r$, and Q_x a probability measure induced on $(\mathbf{C}, \mathcal{M})$ by an SRBM and its attendant pushing process for the data $(S, 0, \Gamma, R)$ and starting point x, we have*

$$E^{Q_x}[\tau_r] \leqq C r^2. \tag{5.18}$$

Proof. This can be proved by the same method as Lemma 6.4 of [37]. The main difference is that we choose $v = \sum_{i \in \mathbf{J}} \lambda_i n_i \in \mathbf{R}^d$ such that $\lambda_i > 0$ and $v \cdot v_i > 0$ for all $i \in \mathbf{J}$. Such a vector v exists by condition **(S.b)**. One then applies Itô's formula to the function $w \to (v \cdot w)^2/2$ and $z(\cdot)$ under Q_x to conclude the desired result in the same manner as in [37]. Note for this that we still have $v'\Gamma v > 0$ since $v \neq 0$.

THEOREM 5.14. *Suppose $F_{\mathbf{J}} = \{0\}$. Then Theorem 1.3 holds.*

Proof. Existence was established previously. The uniqueness can be proved in the same manner as in §6.4 of [37] with the following substitutions. Lemmas 5.1, 5.9, 5.10, 5.12, 5.13, and equation (4.32) should be used in place of Theorem 5.4, Lemmas 6.3, 6.5, 6.7, 6.4, and equation (2.1) of [37]. The Feller continuity and strong Markov property follow by standard arguments (cf. [36, Cor. 4.6]), using the uniqueness established above and the tightness of the laws given by Lemma 5.1.

COROLLARY 5.15. *Suppose that $F_{\mathbf{J}} \neq \varnothing$. Then Theorem 1.3 holds.*

Proof. If the dimension of the vector space spanned by $\{n_i, i \in \mathbf{J}\}$ is less than d, this follows from Theorem 5.3. Otherwise, the dimension of the spanned space is d and $F_{\mathbf{J}}$ must be a single point, which we may take to be the origin. In this case, Theorem 5.14 gives the desired result.

Remark. In the same manner as for the remark following Theorem 5.3, if $F_{\mathbf{J}} \neq \varnothing$ and Z is a stopped SRBM associated with (S, θ, Γ, R) and starting point $x \in S$, where the stopping time T is a stopping time relative to the filtration generated by Z, and Y is the pushing process associated with Z, then the law of $(Z(\cdot \wedge T), Y(\cdot \wedge T))$ is unique.

5.3. Uniqueness in a convex polyhedron.

Proof of Theorem 1.3. Finally, we treat the case of a general convex polyhedron $S \subset \mathbf{R}^d$. The existence in Theorem 1.3 has been established previously. For uniqueness, let Q_x be the probability measure induced on $(\mathbf{C}, \mathcal{M})$ by an SRBM associated with (S, θ, Γ, R) and starting point $x \in S$. We decompose the state space S into subregions, where each subregion is a subset of a cone. We then apply the remark following Corollary 5.15 on each of these cones. More precisely, recall the definitions of C_ε, $F_{\mathbf{K}}^\varepsilon$, $\mathcal{C}$, $S^{\mathbf{K}}$, and $R^{\mathbf{K}}$ from § 4, and for $\varepsilon > 0$ define

$$\begin{aligned} \widetilde{F}_{\mathbf{K}}^\varepsilon = \{x \in \mathbf{R}^d\colon\ & 0 \leqq n_i \cdot x - b_i \leqq 2C_\varepsilon \ \forall i \in \mathbf{K} \text{ and} \\ & n_i \cdot x - b_i > \varepsilon/2 \ \forall i \in \mathbf{J} \backslash \mathbf{K}\}. \end{aligned} \tag{5.19}$$

Order the sets in $\mathcal{C}$. Then let $\sigma_0 = 0$ and define a sequence of pairs $\{(\mathbf{r}_n, \sigma_n)\}_{n=1}^\infty$ by induction as follows. Let $\mathbf{r}_1$ be the first $\mathbf{K} \in \mathcal{C}$ such that $x \in F_{\mathbf{K}}^\varepsilon$ and let $\sigma_1 = \inf\{t \geqq 0\colon\ z(t) \notin \widetilde{F}_{\mathbf{r}_1}^\varepsilon\}$. Assuming $\mathbf{r}_n, \sigma_n$ have been defined, on $\{\sigma_n < \infty\}$, let $\mathbf{r}_{n+1}$ be the first $\mathbf{K} \in \mathcal{C}$ such that $z(\sigma_n) \in F_{\mathbf{K}}^\varepsilon$ and let $\sigma_{n+1} = \inf\{t \geqq \sigma_n\colon\ z(t) \notin \widetilde{F}_{\mathbf{r}_{n+1}}^\varepsilon\}$. On $\{\sigma_n = +\infty\}$, define $\mathbf{r}_{n+1} = \mathbf{r}_n$ and $\sigma_{n+1} = \sigma_n$. By the continuity of the paths of $z(\cdot)$, $\sigma_n \to +\infty$ as $n \to \infty$. Thus, it suffices to prove the uniqueness in law of $(z(\cdot \wedge \sigma_n), y(\cdot \wedge \sigma_n))$ under Q_x for each n. This follows in the same manner as in the proof of Theorem 5.4, except that in place of the remark following Theorem 5.3, one uses the uniqueness of stopped SRBM's with data $(S^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$, where $\mathbf{K} \in \mathcal{C}$ (and hence $F_{\mathbf{K}} \neq \varnothing$) established in the remark following Corollary 5.15.

The Feller continuity and strong Markov property follow in the same manner as for Theorem 5.14.

This completes the induction step and hence Theorem 1.3 holds for all $d \geqq 1$.

We define an SRBM with initial law μ as in Definition 4.2 with $T = \infty$.

COROLLARY 5.16. *For any probability distribution μ on the Borel sets in S, there is an SRBM associated with (S, θ, Γ, R) having initial law μ and this SRBM together with its pushing process Y is unique in law.*

Proof. By the Feller continuity of the probability measures $\{Q_x, x \in S\}$ defined in Theorem 1.3, $P_\mu(\cdot) \equiv \int_S Q_x(\cdot)\mu(dx)$ is well defined as a probability measure on $(\mathbf{C}, \mathcal{M})$. It can be readily verified that the canonical processes $(z(\cdot), y(\cdot))$ under P_μ define an SRBM and pushing process with the desired properties.

For uniqueness one considers the probability measure Q_μ induced on $(\mathbf{C}, \mathcal{M})$ by an SRBM with initial law μ. Then one takes regular conditional probability distributions of Q_μ relative to $\mathcal{M}_0$ to reduce the problem to uniqueness starting from a fixed point in S, and then one applies the uniqueness already established in Theorem 1.3.

Appendix A. Continuity of solutions of martingale problems. Let S be an arbitrary metric space and $\mathcal{A}$ be an algebra of continuous, real-valued functions defined on S and suppose that the constant functions are in $\mathcal{A}$. Let L and D_i, $i = 1, \ldots, m$, be linear operators mapping $\mathcal{A}$ into $\mathcal{A}$. Assume that

(i) Λ, defined by

$$\Lambda(f, g) = \tfrac{1}{2}\{L(fg) - fLg - gLf\} \qquad \text{for all} \quad f, g \in \mathcal{A},$$

is a derivation, i.e.,

$$\Lambda(fg, h) = f\Lambda(g, h) + g\Lambda(f, h) \qquad \text{for all} \quad f,\ g,\ h \in \mathcal{A},$$

(ii) for each $i \in \{1, \ldots, m\}$, D_i satisfies the product rule

$$D_i(fg) = fD_ig + gD_if \qquad \text{for all} \quad f,\ g \in \mathcal{A}.$$

LEMMA A.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space on which are defined processes $Z, Y^0, Y^1, \ldots, Y^m$ such that Z is an adapted process with paths that P-a.s. are in S and are right continuous with finite left limits, and $Y^0, Y^1, \ldots, Y^m$ are adapted processes that are P-a.s. continuous and locally of bounded variation. Suppose that $f \in \mathcal{A}$ and for $g = f,\ f^2,\ f^3,\ f^4$,*

$$M_t^g \equiv g(Z_t) - \int_0^t Lg(Z_s)\,dY_s^0 - \sum_{i=1}^m \int_0^t D_i g(Z_s)\,dY_s^i, \qquad t \geqq 0, \tag{A.1}$$

is an $\{\mathcal{F}_t\}$-local martingale under P. Then $f(Z)$ and M^f are P-a.s. continuous and the quadratic variation process, $[M^f, M^f]_t = 2\int_0^t \Lambda(f,f)(Z_s)\,dY_s^0$ for all $t \geqq 0$.

Remark. This lemma was proved by Bakry and Emery [2, pp. 182–184] for the case where $Y^0(t) \equiv t$ and $Y^i \equiv 0$, $i = 1, \ldots, m$. The generalization stated here follows by minor modification of their proof, as we show below.

Proof. So that we may apply stochastic calculus, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ has been completed. In the following, terms such as almost surely (a.s.) and (local) martingale will be relative to this filtered probability space. By Lemma 2 of Bakry–Emery [2, pp. 182–184], it suffices to show that $H_n(M^f, A^f)$ is a local martingale for $n = 1, 2, 3, 4$, where $A_t^f = 2\int_0^t \Lambda(f,f)(Z_s)\,dY_s^0$ and $H_n(x,a)$ are the Hermite polynomials: $\sum_{n\geqq 0} H_n(x,a)u^n = \exp\left(xu - \frac{1}{2}au^2\right)$ for all $u \in \mathbf{R}$. Note that $\frac{\partial}{\partial x} H_n = H_{n-1}$, $\frac{\partial}{\partial a} H_n = -\frac{1}{2} H_{n-2}$, where $H_n \equiv 0$ for $n < 0$. By an identity for Hermite polynomials we have for $n \geqq 0$,

$$H_n(x+y, a) = \sum_{0 \leqq k \leqq n} \frac{y^k}{k!} H_{n-k}(x,a).$$

Then, by applying (A.1) with $g = f$, we have

$$\begin{aligned} H_n(M_t^f, A_t^f) = \sum_{0\leqq k\leqq n} &\frac{(f(Z_t))^k}{k!} H_{n-k} \\ &\times \left(-\int_0^t Lf(Z_s)dY_s^0 - \sum_{i=1}^m \int_0^t D_i f(Z_s) dY_s^i,\ A_t^f \right). \end{aligned} \tag{A.2}$$

Fix $n \in \{1,2,3,4\}$. Now we have the integration by parts formula $d(WV)_t = V_t\,dW_t + W_t\,dV_t$, where W is a semimartingale with paths a.s. in $D_{\mathbf{R}}[0,\infty)$ and V is a process that is a.s. continuous and locally of bounded variation. By combining this with (A.2) and (A.1) for $d(f(Z_t))^k$, we obtain

$$\begin{aligned} &dH_n(M_t^f, A_t^f) \\ &= \sum_{0\leqq k\leqq n} \frac{1}{k!} \Bigg[H_{n-k}\left(-\int_0^t Lf(Z_s)\,dY_s^0 - \sum_{i=1}^m \int_0^t D_i f(Z_s)\,dY_s^i, A_t^f \right) \\ &\qquad \times \left(dM_t^{f^k} + Lf^k(Z_t)\,dY_t^0 + \sum_{i=1}^m D_i f^k(Z_t)\,dY_t^i \right) \\ &\qquad + f^k(Z_t) \left(-H_{n-k-1} \left(-\int_0^t Lf(Z_s)\,dY_s^0 - \sum_{i=1}^m \int_0^t D_i f(Z_s)\,dY_s^i, A_t^f \right) \right. \end{aligned}$$

$$\times\left(Lf(Z_t)\,dY_t^0+\sum_{i=1}^{m} D_if(Z_t)\,dY_t^i\right)\Bigg)-\frac{1}{2}f^k(Z_t)\,H_{n-k-2}$$
$$\times\left(-\int_0^t Lf(Z_s)\,dY_s^0-\sum_{i=1}^{m}\int_0^t D_if(Z_s)\,dY_s^i,A_t^f\right)dA_t^f\Bigg]$$
$$=\sum_{0\leqq k\leqq n}\frac{1}{k!}\,H_{n-k}(\cdots)\,dM_t^{f^k}$$
$$+\sum_{0\leqq k\leqq n}\frac{1}{k!}\left[H_{n-k}(\cdots)\left(Lf^k(Z_t)\,dY_t^0+\sum_{i=1}^{n} D_if^k(Z_t)\,dY_t^i\right)\right]$$
$$-\sum_{1\leqq k\leqq n}\frac{f^{k-1}(Z_t)}{(k-1)!}\,H_{n-k}(\cdots)\left(Lf(Z_t)\,dY_t^0+\sum_{i=1}^{m} D_if(Z_t)\,dY_t^i\right)$$
$$-\sum_{2\leqq k\leqq n}\frac{f^{k-2}(Z_t)}{(k-2)!}\,H_{n-k}(\cdots)\Lambda(f,f)(Z_t)\,dY_t^0.$$

By the properties (i) and (ii) of $\Lambda, D_1,\ldots,D_m$, we have

$$\begin{aligned}
&Lf^k=kf^{k-1}Lf+k(k-1)f^{k-2}\Lambda(f,f) \qquad \text{for} \quad k\geqq 2,\\
&Lf^k=0 \qquad \text{for} \quad k=0, \qquad Lf^k=Lf \qquad \text{for} \quad k=1,\\
&D_if^k=kf^{k-1}D_if \qquad \text{for} \quad i=1,\ldots,m, \quad k\geqq 1,\\
&D_if^k=0 \quad \text{for} \quad k=0.
\end{aligned}$$

Combining the above yields:

$$dH_n(M_t^f,A_t^f)=\sum_{0\leqq k\leqq n}\frac{1}{k!}\,H_{n-k}(\cdots)\,dM_t^{f^k},$$

and hence $H_n(M^f,A^f)$ is a local martingale. Thus, by Bakry–Emery [2, Lem. 2, p. 182], M^f is P-a.s. continuous with quadratic variation process A^f, and consequently $f(Z)$ is P-a.s. continuous.

Appendix B. Proof of Theorem 3.2. The following lemma plays a critical role in our proof of Theorem 3.2 and is also used elsewhere in the paper, e.g., in § 4. For our purposes, we only need that the constant C depends on S alone. The more general statement of dependence on $\{n_i,\ i\in\mathbf{J}\}$ alone is included here for potential future use in other applications.

LEMMA B.1. *There is a constant $C\geqq 1$ which depends on $\{n_i, i\in\mathbf{J}\}$ such that for each $\mathbf{K}$: $\varnothing\neq\mathbf{K}\subset\mathbf{J}$ and $F_{\mathbf{K}}\neq\varnothing$, and each $x\in S$,*

$$d(x,F_{\mathbf{K}})\leqq C\sum_{i\in\mathbf{K}}(n_i\cdot x-b_i). \tag{B.1}$$

Proof. For fixed $\mathbf{K}$ such that $\varnothing\neq\mathbf{K}\subset\mathbf{J}$ and $F_{\mathbf{K}}\neq\varnothing$, the estimate (B.1) follows from a theorem of Hoffman [25], with supporting lemmas proved by Agmon [1]. To see how Hoffman's theorem applies one needs to let his matrix A have rows given by $\{-n_i,\ i\in\mathbf{J};\ n_i,\ i\in\mathbf{K}\}$ and his b have entries $\{-b_i,\ i\in\mathbf{J};\ b_i,\ i\in\mathbf{K}\}$. His functions F_m, F_n can be taken to equal the usual Euclidean distance functions on $\mathbf{R}^m$, $\mathbf{R}^n$, respectively. Examination of Hoffman's proof reveals that the constant C

can be chosen to depend only on the rows of A (and not on b). Since there are only finitely many $\mathbf{K} \subset \mathbf{J}$, the constant C can be chosen to depend only on $\{n_i, i \in \mathbf{J}\}$.

Proof of Theorem 3.2. We shall use an induction procedure to define a function $g_m \in C_b^2(S)$ satisfying $D_i g_m \geqq 1$ on F_i for all $i \in \mathbf{J} = \{1, \ldots, m\}$. First we need some definitions.

From condition **(S.b)**, for each maximal $\mathbf{K} \subset \mathbf{J}$, there are $c_i^{\mathbf{K}} > 0$ for $i \in \mathbf{K}$ such that $\eta^{\mathbf{K}} \equiv \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} n_i$ satisfies

$$\eta^{\mathbf{K}} \cdot v_i > 0 \qquad \text{for all} \quad i \in \mathbf{K}. \tag{B.2}$$

Without loss of generality, we may and do assume that

$$\sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} = \frac{1}{2}. \tag{B.3}$$

Let

$$c = \min\{c_i^{\mathbf{K}} \colon i \in \mathbf{K},\ \mathbf{K} \text{ is maximal}\},$$
$$\delta = \min\{n_i \cdot x - b_i \colon x \in F_{\mathbf{K}},\ F_i \cap F_{\mathbf{K}} = \varnothing,\ \mathbf{K} \text{ is maximal}\}.$$

Note that by (B.3), $c \leqq \frac{1}{2}$. Since $|\mathbf{K}| \leqq m$ for all $\mathbf{K} \subset \mathbf{J}$, by Lemma B.1, there is a constant $C \geqq 1$ such that

$$d(x, F_{\mathbf{K}}) \leqq C \max_{i \in \mathbf{K}} (n_i \cdot x - b_i), \tag{B.4}$$

for all $x \in S$, $\mathbf{K}$: $\varnothing \not\equiv \mathbf{K} \subset \mathbf{J}$ and $F_{\mathbf{K}} \neq \varnothing$. Define a sequence $\{(\gamma_k, \beta_k),\ k = 0, 1, \ldots, m\}$ as follows:

$$\begin{aligned}
\gamma_m &= \frac{\delta}{2(C+1)}, & \beta_m &= c\gamma_m, \\
\gamma_{m-1} &= \frac{\beta_m}{C}, & \beta_{m-1} &= c\gamma_{m-1}, \\
&\vdots \\
\gamma_0 &= \frac{\beta_1}{C}, & \beta_0 &= c\gamma_0.
\end{aligned}$$

Note that $\gamma_k/\beta_k = 1/c \geqq 2$, $\beta_k/\gamma_{k-1} = C \geqq 1$ for all k for which these expressions are defined. Let $\mathcal{M}_0$ denote the collection of all maximal $\mathbf{K} \subset \mathbf{J}$, together with the empty set. For each $\mathbf{K} \in \mathcal{M}_0$ and $0 \leqq k \leqq m$, let

$$B^{k,\mathbf{K}} = \{x \in S \colon n_i \cdot x - b_i \geqq \gamma_k \ \text{ for each } \ i \notin \mathbf{K}\},$$
$$B_{k,\mathbf{K}} = \{x \in S \colon n_i \cdot x - b_i \leqq \beta_k \ \text{ for each } \ i \in \mathbf{K}\}.$$

Let φ and ψ be twice continuously differentiable, nondecreasing functions defined from $\mathbf{R}_+$ into $\mathbf{R}_+$ such that

$$\varphi(x) = \begin{cases} x - 1 & \text{for} \quad 0 \leqq x \leqq \frac{1}{2}, \\ 0 & \text{for} \quad x \geqq 1, \end{cases}$$
$$\psi(x) = \begin{cases} 0 & \text{for} \quad 0 \leqq x \leqq \frac{1}{2}, \\ 1 & \text{for} \quad x \geqq 1. \end{cases}$$

We now begin the induction proof. Let $g_0 = 0$ and $M_0 = 0$. Note that $D_i g_0 \geqq -M_0$ on F_i and $F_i \cap B^{0,\varnothing} = \varnothing$, $\forall i \in \mathbf{J}$. Fix $1 \leqq k \leqq m$ and suppose that $g_{k-1} \in C_b^2(S)$ and $M_{k-1} \geqq 0$ have been defined such that for each $i \in \mathbf{J}$:

$$D_i g_{k-1} \geqq -M_{k-1} \quad \text{on} \quad F_i, \tag{B.5}$$

$$D_i g_{k-1} \geqq 1 \quad \text{on} \quad F_i \cap B^{k-1,\mathbf{K}} \quad \text{for all maximal} \quad \mathbf{K}\colon |\mathbf{K}| \leqq k-1 \quad \text{and} \quad i \in \mathbf{K}. \tag{B.6}$$

Note that if $i \notin \mathbf{K}$, then $F_i \cap B^{j,\mathbf{K}} = \varnothing$ for $0 \leqq j \leqq m$.

For each maximal $\mathbf{K} \subset \mathbf{J}$ satisfying $|\mathbf{K}| = k$, let $a_{\mathbf{K}} > 0$:

$$a_{\mathbf{K}} \eta^{\mathbf{K}} \cdot v_i \geqq \beta_k (M_{k-1} + 1) \qquad \text{for all} \quad i \in \mathbf{K}, \tag{B.7}$$

and define

$$f_{\mathbf{K}}(x) = a_{\mathbf{K}}\, \varphi\left(\frac{\eta^{\mathbf{K}} \cdot (x - x^{\mathbf{K}})}{\beta_k}\right) \prod_{j \notin \mathbf{K}} \psi\left(\frac{n_j \cdot x - b_j}{\gamma_k}\right), \qquad x \in S, \tag{B.8}$$

where $x^{\mathbf{K}}$ is a fixed but arbitrary point in $F_{\mathbf{K}}$. Note that $x^{\mathbf{K}} \in F_{\mathbf{K}}$ implies that $n_i \cdot x^{\mathbf{K}} = b_i$ for all $i \in \mathbf{K}$. Observe that $f_{\mathbf{K}} \in C_b^2(S)$. For $i \notin \mathbf{K}$,

$$D_i f_{\mathbf{K}} = 0 \;\text{ on }\; F_i,$$

since for $x \in F_i$, $n_i \cdot x = b_i$ and then $\psi((n_i \cdot x - b_i)/\gamma_k) = 0$, $\psi'((n_i \cdot x - b_i)/\gamma_k) = 0$. For $i \in \mathbf{K}$,

$$D_i f_{\mathbf{K}}(x) = \begin{cases} \dfrac{a_{\mathbf{K}}\, \eta^{\mathbf{K}} \cdot v_i}{\beta_k}\, \varphi'\left(\dfrac{\eta^{\mathbf{K}} \cdot (x - x^{\mathbf{K}})}{\beta_k}\right) \geqq 0, & \\ \qquad\qquad x \in F_i \cap B^{k,\mathbf{K}}, & \\ \dfrac{a_{\mathbf{K}}\, \eta^{\mathbf{K}} \cdot v_i}{\beta_k}\, \varphi'\left(\dfrac{\eta^{\mathbf{K}} \cdot (x - x^{\mathbf{K}})}{\beta_k}\right) \geqq M_{k-1} + 1, & \\ \qquad\qquad x \in F_i \cap B^{k,\mathbf{K}} \cap B_{k,\mathbf{K}}, & \\ 0, & \begin{cases} x \in F_i \cap B^{k,\widetilde{\mathbf{K}}} \text{ and} \\ \widetilde{\mathbf{K}} \in \mathcal{M}_0\colon |\widetilde{\mathbf{K}}| \leqq k, \\ \widetilde{\mathbf{K}} \neq \mathbf{K}. \end{cases} \end{cases} \tag{B.9}$$

For the two inequalities above, note that

$$\begin{aligned} \eta^{\mathbf{K}} \cdot (x - x^{\mathbf{K}}) &= \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} n_i \cdot (x - x^{\mathbf{K}}) \\ &= \begin{cases} \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} (n_i \cdot x - b_i) \geqq 0 & \forall x \in S, \\ \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} (n_i \cdot x - b_i) \leqq \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} \beta_k = \frac{\beta_k}{2} & \forall x \in B_{k,\mathbf{K}}. \end{cases} \end{aligned}$$

For the last equality in (B.9), note that if $x \in F_i \cap B^{k,\widetilde{\mathbf{K}}}$ for some $\widetilde{\mathbf{K}} \in \mathcal{M}_0$: $|\widetilde{\mathbf{K}}| \leqq k$, $\widetilde{\mathbf{K}} \neq \mathbf{K}$, there is $j \in \mathbf{K} \cap \widetilde{\mathbf{K}}^c$ and so $n_j \cdot x - b_j \geqq \gamma_k$ which implies

$$\eta^{\mathbf{K}} \cdot (x - x^{\mathbf{K}}) = \sum_{i \in \mathbf{K}} c_i^{\mathbf{K}} (n_i \cdot x - b_i) \geqq c_j^{\mathbf{K}} \gamma_k \geqq c\gamma_k = \beta_k,$$

and hence

$$\varphi\left(\frac{\eta^{\mathbf{K}}\cdot(x-x^{\mathbf{K}})}{\beta_k}\right)=0,\qquad \varphi'\left(\frac{\eta^{\mathbf{K}}\cdot(x-x^{\mathbf{K}})}{\beta_k}\right)=0.$$

Define

$$M_k = M_{k-1}+\sum{}'\Big(-\inf_{x\in F_i,\, i\in\mathbf{K}} D_i f_{\mathbf{K}}(x)\Big)^+, \tag{B.10}$$

$$g_k = g_{k-1}+\sum{}' f_{\mathbf{K}}, \tag{B.11}$$

where $\sum'$ is summation over all maximal $\mathbf{K}$ with $|\mathbf{K}|=k$. Then for each $i\in\mathbf{J}$,

$$D_i g_k \geqq -M_k \quad \text{on} \quad F_i. \tag{B.12}$$

Fix $\widetilde{\mathbf{K}}$ maximal: $|\widetilde{\mathbf{K}}|\leqq k$, $i\in\widetilde{\mathbf{K}}$. Then on $F_i\cap B^{k,\widetilde{\mathbf{K}}}$,

$$D_i g_k = \begin{cases} D_i g_{k-1}+D_i f_{\widetilde{\mathbf{K}}}+\sum'' D_i f_{\mathbf{K}}, & \text{if} \quad |\widetilde{\mathbf{K}}|=k,\\ D_i g_{k-1}+\sum''' D_i f_{\mathbf{K}}, & \text{if} \quad |\widetilde{\mathbf{K}}|\leqq k-1,\end{cases}$$

where $\sum''$ is summation over all maximal $\mathbf{K}$ such that $|\mathbf{K}|=k$, $i\in\mathbf{K}$, $\mathbf{K}\neq\widetilde{\mathbf{K}}$, and $\sum'''$ is summation over all maximal $\mathbf{K}$ such that $|\mathbf{K}|=k$, $i\in\mathbf{K}$.

We claim that the right member above is greater than or equal to one. This can be seen as follows. For the case $|\widetilde{\mathbf{K}}|\leqq k-1$, this follows from the induction assumption (B.6), the fact that $B^{k,\widetilde{\mathbf{K}}}\subset B^{k-1,\widetilde{\mathbf{K}}}$, and (B.9). For the case $|\widetilde{\mathbf{K}}|=k$, we consider two possibilities. First, if $x\in F_i\cap B^{k,\widetilde{\mathbf{K}}}\cap B_{k,\widetilde{\mathbf{K}}}$, then $D_i g_k(x)\geqq -M_{k-1}+(M_{k-1}+1)+0$ by (B.5), (B.9). On the other hand, if $x\in F_i\cap B^{k,\widetilde{\mathbf{K}}}\cap B^c_{k,\widetilde{\mathbf{K}}}$, we claim

$$x\in \bigcup_{\substack{\mathbf{L}\ \text{maximal}\\ |\mathbf{L}|\leqq k-1,\ i\in\mathbf{L}}} B^{k-1,\mathbf{L}}. \tag{B.13}$$

This can be proved as follows. For $x\in F_i\cap B^{k,\widetilde{\mathbf{K}}}\cap B^c_{k,\widetilde{\mathbf{K}}}$,

$$n_j\cdot x-b_j\geqq \gamma_k \qquad \text{for all} \quad j\notin\widetilde{\mathbf{K}},$$

and for some $j_0\in\widetilde{\mathbf{K}}$,

$$n_{j_0}\cdot x-b_{j_0}>\beta_k.$$

Thus, $x\in F_{\mathbf{M}}$ for some $\mathbf{M}\subset\mathbf{J}$: $|\mathbf{M}|<k$, $\mathbf{M}$ is maximal, $i\in\mathbf{M}$. If $x\in B^{k-1,\mathbf{M}}$ then (B.13) holds. If $x\notin B^{k-1,\mathbf{M}}$, there is $j_1\notin\mathbf{M}$: $n_{j_1}\cdot x-b_{j_1}<\gamma_{k-1}$. Note from the definition of δ that F_{j_1} must meet $F_{\mathbf{M}}$, because $x\in F_{\mathbf{M}}$ and $n_{j_1}\cdot x-b_{j_1}<\delta$. Let $\mathbf{M}'\subset\mathbf{J}$ be maximal such that

$$\mathbf{M}'\supset\mathbf{M}\cup\{j_1\},\quad F_{\mathbf{M}'}=F_{\mathbf{M}\cup\{j_1\}}.$$

For each $j\in\mathbf{M}'$,

$$\begin{aligned} n_j\cdot x-b_j &\leqq d(x,F_{\mathbf{M}'})=d(x,F_{\mathbf{M}\cup\{j_1\}})\\ &\leqq C\max_{\ell\in\mathbf{M}\cup\{j_1\}}(n_\ell\cdot x-b_\ell)<C\gamma_{k-1}=\beta_k<\gamma_k.\end{aligned} \tag{B.14}$$

It follows that $\mathbf{M}'$ does not meet $\widetilde{\mathbf{K}}^c \cup \{j_0\}$ and so $|\mathbf{M}'| \leqq k-1$. If $x \in B^{k-1,\mathbf{M}'}$, (B.13) holds. If not, there is $j_2 \notin \mathbf{M}'$ such that $n_{j_2} \cdot x - b_{j_2} < \gamma_{k-1}$. Let $\overline{x} \in F_{\mathbf{M}'}$ such that $d(x, F_{\mathbf{M}'}) = |x - \overline{x}|$. Then

$$\begin{aligned} n_{j_2} \cdot \bar{x} - b_{j_2} &= n_{j_2} \cdot (\bar{x} - x) + n_{j_2} \cdot x - b_{j_2} \\ &\leqq |\bar{x} - x| + \gamma_{k-1} = d(x, F_{\mathbf{M}'}) + \gamma_{k-1} \leqq \beta_k + \gamma_{k-1} < \gamma_k < \delta. \end{aligned} \tag{B.15}$$

Thus, F_{j_2} must meet $F_{\mathbf{M}'}$. Let $\mathbf{M}'' \subset \mathbf{J}$ be maximal such that

$$\mathbf{M}'' \supset \mathbf{M}' \cup \{j_2\}, \quad F_{\mathbf{M}''} = F_{\mathbf{M}' \cup \{j_2\}} = F_{\mathbf{M} \cup \{j_1, j_2\}}.$$

Continuing in a similar manner to that from (B.14) onward, we must eventually get $x \in B^{k-1,\mathbf{L}}$ for some maximal $\mathbf{L}$ satisfying $|\mathbf{L}| \leqq k-1$, $i \in \mathbf{L}$, since we augment the index set by at least one at each stage. Then we see that for $|\widetilde{\mathbf{K}}| = k$, $x \in F_i \cap B^{k,\widetilde{\mathbf{K}}} \cap B^c_{k,\widetilde{\mathbf{K}}}$, $D_i g_k \geqq 1 + 0 + 0$ by (B.13), (B.6), (B.9). Thus,

$$D_i g_k \geqq 1 \quad \text{on} \quad F_i \cap B^{k,\widetilde{\mathbf{K}}} \quad \text{for all} \quad \widetilde{\mathbf{K}} \text{ maximal: } |\widetilde{\mathbf{K}}| \leqq k, \ i \in \widetilde{\mathbf{K}}.$$

This completes the induction step and so there is $g_m \in C_b^2(S)$ such that for each $i \in \mathbf{J}$,

$$D_i g_m \geqq 1 \quad \text{on} \quad F_i \cap B^{m,\mathbf{K}} \quad \text{for all maximal } \mathbf{K}\text{: } |\mathbf{K}| \leqq m, \ i \in \mathbf{K}.$$

The proof is completed by observing that for each $x \in F_i$ there is a maximal set $\mathbf{L}$ such that $|\mathbf{L}| \leqq m$, $i \in \mathbf{L}$, and $x \in F_i \cap B^{m,\mathbf{L}}$. The last statement can be proved in a similar manner to (B.13) as follows. Fix $x \in F_i$. Let $\mathbf{M}$ be a maximal set containing i such that $x \in F_{\mathbf{M}}$. If $x \notin B^{m,\mathbf{M}}$, then there is $j_1 \notin \mathbf{M}$ such that $n_{j_1} \cdot x - b_{j_1} < \gamma_m < \delta/2$. Then F_{j_1} must meet $F_{\mathbf{M}}$. Let $\mathbf{M}'$ be maximal such that $\mathbf{M}' \supset \mathbf{M} \cup \{j_1\}$ and $F_{\mathbf{M}'} = F_{\mathbf{M} \cup \{j_1\}}$. If $x \in B^{m,\mathbf{M}'}$, we are finished. If $x \notin B^{m,\mathbf{M}'}$, then we proceed in a similar manner to that for $x \notin B^{m,\mathbf{M}}$. In particular, one has to do a calculation similar to that in (B.15) using the fact that $d(x, F_{\mathbf{M}'}) \leqq C\gamma_m < \delta/2$. Continuing in this manner, incrementing the index set by at least one at each stage, we must eventually obtain a maximal set with the desired properties.

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