*The Annals of Applied Probability*
1992, Vol. 2, No. 1, 65–86

# REFLECTED BROWNIAN MOTION IN AN ORTHANT: NUMERICAL METHODS FOR STEADY-STATE ANALYSIS

BY J. G. DAI AND J. M. HARRISON

*Georgia Institute of Technology and Stanford University*

This paper is concerned with a class of multidimensional diffusion processes, variously known as reflected Brownian motions, regulated Brownian motions, or just RBM's, that arise as approximate models of queueing networks. We develop an algorithm for numerical analysis of a semimartingale RBM with state space $S = \mathbb{R}_+^d$ (the nonnegative orthant of $d$-dimensional Euclidean space). This algorithm lies at the heart of the QNET method for approximate two-moment analysis of open queueing networks.

## Contents



**1. Introduction.** This paper is concerned with a class of multidimensional diffusion processes, variously known as reflected Brownian motions, regulated Brownian motions, or just RBM's, that arise as approximate models of queueing networks. More specifically, we develop methods for numerical analysis of a semimartingale RBM (SRBM) with state space $S = \mathbb{R}_+^d$ (the nonnegative orthant of $d$-dimensional Euclidean space). On the interior of $S$ this process, denoted by $Z = \{Z(t),\ t \geq 0\}$, behaves like an ordinary Brownian motion with drift vector $\mu$ and covariance matrix $\Gamma$, and on each of the $(d-1)$-dimensional hyperplanes that form the boundary of $S$, it is instantaneously reflected in a direction that is constant over that boundary face. The precise mathematical definition of the process $Z$ will be given later in Section 2.

The motivation for our study comes from the theory of open queueing networks, that is, networks of interacting processors or service stations where customers arrive from outside the network, visit one or more stations in an order that may vary from one customer to the next and then depart. (In

Received August 1990; revised February 1991,
*AMS* 1980 *subject classifications*. Primary 60J70, 60K30, 65U05; secondary 65P05, 68M20.
*Key words and phrases*. Brownian system model, reflected Brownian motion, stationary distribution, numerical analysis, open queueing networks, performance analysis

contrast, a closed queueing network is one where a fixed customer population circulates perpetually through the stations of the network, with no new arrivals and no departures.) It was shown by Reiman [19] that the $d$-dimensional queue length process associated with a certain type of open $d$-station network, if properly normalized, converges under heavy traffic conditions to a corresponding SRBM with state space $\mathbb{R}_+^d$. Peterson [18] proved a similar heavy traffic limit theorem for open queueing networks with multiple customer types and deterministic feedforward customer routing; Peterson's assumptions concerning the statistical distribution of customer routes are in some ways more general and in some ways more restrictive than Reiman's. The upshot of this work on limit theorems was to show that SRBM's with state space $\mathbb{R}_+^d$ may serve as good approximations, at least under heavy traffic conditions, for the queue length processes, workload processes and waiting time processes associated with various types of open $d$-station networks. Recently Harrison and Nguyen [12] have defined a very general class of open queueing networks and articulated a systematic procedure for approximating the associated stochastic processes by SRBM's. This general approximation scheme subsumes those suggested by the limit theorems of both Reiman and Peterson, but it has not yet been buttressed by a rigorous and equally general heavy traffic limit theory.

In light of the work described above, one may take the point of view that an SRBM with state space $\mathbb{R}_+^d$ represents a diffusion model or Brownian model of a $d$-station open queueing network, at least if the data of the SRBM are correctly chosen, and that this Brownian model is an alternative to the more familiar models emphasized in conventional queueing theory. If the replacement of a conventional queueing model by its Brownian analog is to yield benefits, of course, one must be able to compute interesting performance measures for the Brownian model, and it is steady-state performance measures that are usually of greatest interest in queueing theory. Thus we are led to the problem of computing the stationary distribution of an SRBM in an orthant, or at least computing summary statistics of the stationary distribution. As we will explain later, the stationary distribution is the solution of a certain highly structured partial differential equation problem (PDE problem). The purpose of this paper is to develop a general computational method for numerical solution of that PDE problem, prove some properties of the general method and provide some (admittedly incomplete) evidence as to its practical efficiency. The method developed here is closely related to one described in an earlier paper [3] for steady-state analysis of an SRBM in a two-dimensional rectangle; however, in the orthant setting there are new issues to be dealt with because of the unbounded state space.

The paper is organized as follows. Section 2 gives a precise mathematical definition of the SRBM to be studied and Section 3 develops the analytical characterization of its stationary distribution. A general method for computing the stationary distribution is presented in Section 4 and Section 5 describes the choices that we have made in implementing the general method. (Readers will see that other choices are certainly possible.) In Section 6 we consider a

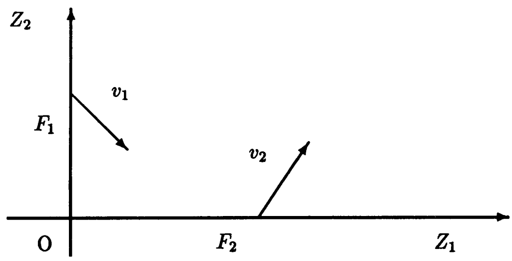


FIG. 1. *The state space $S$ when $d = 2$.*

number of test problems, comparing the numerical results obtained with our algorithm against known exact results. Finally, Section 7 presents a simple example of an open queueing network that has been studied previously in the literature of computer science: we develop the data of the SRBM, use our algorithm to compute the approximate steady-state performance measures and compare those figures against simulation results and other approximate performance estimates that have been proposed elsewhere. This last section gives additional evidence that our algorithm works and also gives a concrete example of the algorithm's role in performance analysis of queueing systems.

**2. Definitions and preliminaries.** Let $d \geq 1$ be an integer and $S \equiv \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d\colon x_i \geq 0,\ i = 1, 2, \dots, d\}$ be the nonnegative orthant in $d$-dimensional Euclidean space $\mathbb{R}^d$. For $i = 1, 2, \dots, d$, let $F_i \equiv \{x \in S\colon x_i = 0\}$ be the $i$th face of $\partial S$ and $v_i$ be a vector on $F_i$ pointing into $S$ (that is, the $i$th component of $v_i$ is strictly positive). Figure 1 shows an example with $d = 2$. Consider $R \equiv (v_1, v_2, \dots, v_d)$ as a $d \times d$ matrix and let $\Gamma$ be a $d \times d$ symmetric positive definite matrix and $\mu$ a $d$-dimensional vector. For the purpose of this paper we introduce a sample path space

$$C_S \equiv \{\text{functions } w\colon [0, \infty) \to S, \text{ such that } w \text{ is continuous}\}.$$

The *canonical process* on $C_S$ is $Z = \{Z(t, \cdot\,), t \geq 0\}$ defined by

$$Z(t, w) \equiv w(t) \quad \text{for } w \in C_S,\ t \geq 0.$$

The natural filtration associated with $C_S$ is $\{\mathscr{M}_t\}$, where $\mathscr{M}_t \equiv \sigma\{Z(s)\colon 0 \leq s \leq t\}$, $t \geq 0$. For $t \geq 0$, $\mathscr{M}_t$ can also be characterized as the smallest $\sigma$-algebra of subsets of $C_S$ which makes $Z(s)$ measurable for each $0 \leq s \leq t$. The natural $\sigma$-algebra associated with $C_S$ is $\mathscr{M} \equiv \sigma\{Z(s)\colon 0 \leq s < \infty\} = \bigvee_{t=0}^{\infty} \mathscr{M}_t$. ($\bigvee_{t=0}^{\infty} \mathscr{M}_t$ is defined to be the smallest $\sigma$-algebra containing $\mathscr{M}_t$ for each $t \geq 0$.)

In this paper we are concerned with the class of *semimartingale reflected Brownian motions* in $S$, defined as follows.

DEFINITION 1. Let $Z$ be the canonical process defined above and let $\{P_x, x \in S\}$ be a family of probability measures defined on the filtered probability space $(C_S, \mathscr{M}, \{\mathscr{M}_t\})$. We say that $Z$ with $\{P_x, x \in S\}$ is a semimartingale reflected Brownian motion (SRBM) associated with data $(S, \Gamma, \mu, R)$ if for each $x \in S$ we have, $P_x$-almost surely,

$$Z(t) = X(t) + RL(t) = X(t) + \sum_{i=1}^{d} L_i(t) \cdot v_i, \qquad t \geq 0, \tag{1}$$

(2) $X(0) = x$ and $X$ is a $d$-dimensional Brownian motion with covariance matrix $\Gamma$ and drift vector $\mu$ such that $\{X(t) - \mu t, \mathscr{M}_t, t \geq 0\}$ is a martingale,

(3) $L$ is a continuous $\{\mathscr{M}_t\}$-adapted $d$-dimensional process such that $L(0) = 0$, $L$ is nondecreasing and $L_i$ can increase only at times $t$ when $Z_i(t) = 0$ $(i = 1, \dots, d)$.

REMARK. This is actually a special case of the definition advanced by Reiman and Williams [20], who define an SRBM on an arbitrary filtered probability space. Thus one might say that we are dealing with a canonical SRBM. Reiman and Williams pointed out that $\{X(t) - \mu t\}$ being an $\{\mathscr{M}_t\}$-martingale is necessary for an SRBM to have certain desired properties.

An SRBM $Z$ as defined above behaves in the interior of its state space like a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $\Gamma$. When the boundary face $F_i$ is hit, the process $L_i$ (sometimes called the local time of $Z$ on $F_i$) increases, causing an instantaneous displacement of $Z$ in the direction given by $v_i$; the magnitude of the displacement is the minimal amount required to keep $Z$ always inside $S$. Therefore, we call $\Gamma$, $\mu$ and $R$ the covariance matrix, the drift vector and the reflection matrix of $Z$, respectively.

DEFINITION 2. We say that the SRBM in Definition 1 is unique if the corresponding family of probability measures $\{P_x, x \in S\}$ is unique.

DEFINITION 3. A $d \times d$ matrix $A$ is said to be an $\mathscr{S}$ matrix if there exists a $d$-dimensional vector $u \geq 0$ such that $Au > 0$, and to be a completely-$\mathscr{S}$ matrix if each of its principal submatrices is an $\mathscr{S}$ matrix.

Reiman and Williams [20] proved that the reflection matrix $R$ being completely-$\mathscr{S}$ is a necessary condition for existence of an SRBM $Z$ as defined above. The following fundamental problem was recently resolved by Taylor and Williams [21]. For an alternative proof of Feller continuity, see Section 3.2 of Dai [2].

PROPOSITION 1. *Let there be given a covariance matrix $\Gamma$, a drift vector $\mu$ and a reflection matrix $R$. Assume $R$ is completely-$\mathscr{S}$. Then there is a unique SRBM $Z$ (with associated $\{P_x, x \in S\}$) associated with the data $(S, \Gamma, \mu, R)$. Moreover, $Z$ together with $\{P_x, x \in S\}$ is a strong Markov process and Feller continuous, that is, $x \to E^{P_x}[f(Z(t))]$ is continuous for all $f \in C_b(S)$ and $t \geq 0$, where $C_b(S)$ is the set of bounded continuous functions on $S$.*

Hereafter we assume the reflection matrix $R$ is completely-$\mathscr{S}$ and use $\{P_x, x \in S\}$ to denote the unique family which makes $Z$ an SRBM. For each $x \in S$, let $E_x$ denote the expectation operator under $P_x$. For a probability measure $\pi$ on $S$, let

$$P_\pi(\cdot) \equiv \int_S P_x(\cdot)\pi(dx) \tag{4}$$

and $E_\pi$ be the corresponding expectation operator. [The integral in (4) is well defined because of the Feller continuity and Markov property of $\{P_x, x \in S\}$.]

DEFINITION 4. A probability measure $\pi$ on $S$ is called a stationary distribution for the SRBM $Z$ if for every bounded Borel function $f$ on $S$ and every $t > 0$,

$$\int_S E_x[f(Z(t))]\pi(dx) = \int_S f(x)\pi(dx).$$

The following proposition lists some of the properties of the stationary distribution of an SRBM.

PROPOSITION 2. *Let $Z$ be an SRBM associated with data $(S, \Gamma, \mu, R)$. Assume there exists a stationary distribution for $Z$. Then*:

(a) *the stationary distribution $\pi$ of $Z$ is unique and it is absolutely continuous with respect to the Lebesgue measure $dx$*;

(b) *$Z$ is ergodic and for any $f \in C_b(S)$ and $x \in S$,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E_x f(Z(s))\, ds = \int_S f(x)\pi(dx);$$

(c) *$R$ is invertible.*

PROOF. The proofs of parts (a) and (b) are essentially the same as the proof of Theorem 7.1 of Harrison and Williams [13], where the authors considered SRBM's with reflection matrix $R$ being Minkowski. ($R$ is said to be Minkowski if all the elements of $I - R$ are nonnegative and $I - R$ is transient, that is, all the eigenvalues of $I - R$ are strictly less than 1.) Therefore, we only give the proof of part (c).

Assume $R$ is singular. Then there exists a nontrivial vector $v$ such that $v'R = 0$, where prime is the transpose operator. For the SRBM $Z$, we have the semimartingale representation (1). Therefore

$$v'Z(t) = v'X(t) + v'RL(t) = v'X(t), \tag{5}$$

because $v'R = 0$. From part (b), $Z$ is ergodic and hence $v'Z$ is ergodic. On the other hand, $v'X$ is a one-dimensional $(v'\Gamma v, v'\mu)$-Brownian motion which cannot be ergodic, contradicting (5). Thus $R$ cannot be singular. This proves part (c). □

## 3. The basic adjoint relationship.

PROPOSITION 3. *Let $\pi$ be the stationary distribution for an $(S, \Gamma, \mu, R)$-SRBM $Z$ and $\mathscr{B}_{F_i}$ be the Borel $\sigma$-field of $F_i$ $(i = 1, \dots, d)$. For each $i = 1, \dots, d$, there exists a finite Borel measure $\nu_i$ on $F_i$ such that*

$$E_\pi\left\{\int_0^t 1_A(Z_s)\, dL_i(s)\right\} = \tfrac{1}{2} t \nu_i(A), \qquad t \geq 0,\ A \in \mathscr{B}_{F_i}, \tag{6}$$

*and $\nu_i$ is absolutely continuous with respect to the $(d-1)$-dimensional Lebesgue measure $d\sigma_i$ on $F_i$ $(i = 1, 2, \dots, d)$. Furthermore, denoting $d\pi/dx \equiv p_0$ and $d\nu_i/d\sigma_i \equiv p_i$ $(i = 1, \dots, d)$, $p \equiv (p_0; p_1, \dots, p_d)$ jointly satisfy the following basic adjoint relationship:*

$$\int_S (\mathscr{G} f \cdot p_0)\, dx + \tfrac{1}{2} \sum_{i=1}^d \int_{F_i} (\mathscr{D}_i f \cdot p_i)\, d\sigma_i = 0 \quad \textit{for all } f \in C_b^2(S), \tag{7}$$

*where*

$$\mathscr{G} f \equiv \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Gamma_{ij} \frac{\partial^2 f}{\partial x_i\, \partial x_j} + \sum_{i=1}^d \mu_i \frac{\partial f}{\partial x_i}, \tag{8}$$

$$\mathscr{D}_i f(x) \equiv v_i \cdot \nabla f(x) \quad \textit{for } x \in F_i\ (i = 1, 2, \dots, d), \tag{9}$$

*and $C_b^2(S)$ is the space of twice differentiable functions which together with their first and second order partials are continuous and bounded on $S$.*

PROOF. The reflection matrix $R$ is a completely-$\mathscr{S}$ matrix and therefore, in the terminology of Bernard and El Kharroubi [1] it is completely *saillante*. Then Lemma 1 of [1] implies that for some constant $C$,

$$L_i(t) \leq C \max_{0 \leq s \leq t} |X(s) - X(0)|, \qquad P_x\text{-a.s., } x \in S.$$

Because $X - X(0)$ is a $(\Gamma, \mu)$-Brownian motion starting from the origin under each $P_x$, we have

$$\sup_{x \in S} E_x[L_i(t)] < \infty, \qquad t \geq 0\ (i = 1, 2, \dots, d).$$

The rest of the proof follows exactly the proof of Theorem 8.1 in [13] when $R$ is Minkowski. □

Proposition 3 shows that the basic adjoint relationship (7) is necessary for

$$p = (p_0; p_1, \dots, p_d)$$

to be the stationary density of $Z$. The following essential complement, which was conjectured in [13], was recently proved by Dai and Kurtz [4].

PROPOSITION 4. *Suppose that $p_0$ is a probability density function on $S$ and $p_i$ is an integrable (with respect to $d\sigma_i$) nonnegative Borel function on $F_i$ $(i = 1, \dots, d)$. If $p_0$ together with $p_1, \dots, p_d$ jointly satisfy the basic adjoint* *relationship* (7), *then* $p_0$ *is the stationary density of* $Z$ *and* $\nu_i$ *defined by* $d\nu_i \equiv p_i\, d\sigma_i$ $(i = 1, 2, \ldots, d)$ *is the boundary measure defined in* (6) *above.*

Here we conjecture that a stronger result is true.

CONJECTURE 1. *Suppose that* $p_0$ *is an integrable Borel function on* $S$ *and* $p_1, \ldots, p_d$ *are integrable Borel functions on* $F_1, \ldots, F_d$, *respectively. If* $\int_S p_0(x)\,dx = 1$ *and* $(p_0; p_1, \ldots, p_d)$ *jointly satisfy the basic adjoint relationship* (7), *then* $p_i$ *is nonnegative* $(i = 0, 1, 2, \ldots, d)$.

REMARK. The conjecture simply says that if $p = (p_0; p_1, \ldots, p_d)$ satisfies the basic adjoint relationship (7), then $p$ does not change sign.

Hereafter, we simply denote the stationary density of the SRBM by $p$. As explained in the introductory section, our primary task in this paper is to compute the stationary density $p$. For a driftless reflected Brownian motion in a bounded two-dimensional region the work of Harrison, Landau and Shepp [15] gives an analytical expression for the stationary density, and the availability of a package for computation of Schwartz–Christoffel transformations makes evaluation of the associated performance measures numerically feasible; see [22]. For the two-dimensional case with drift, Foddy [7] found analytical expressions for the stationary densities for certain special domains, drifts and directions of reflection, using Riemann–Hilbert techniques. In dimensions three and more, SRBM's having stationary distributions of exponential form were identified in [14, 24] and these results were applied in [13, 16] to SRBM's arising as approximations to open and closed queueing networks with homogeneous customer populations. However, until now there has been no general method for solving the PDE problems in (7) and it is very unlikely that a general analytical expression for $p$ can ever be found.

In the next section we will develop an algorithm for computing a solution $p$ to the basic adjoint relationship (7). We end this section by converting (7) into a compact form that will be used in the next section. Given an $f \in C_b^2(S)$, let

$$\mathscr{A}f \equiv (\mathscr{G}f; \mathscr{D}_1 f, \ldots, \mathscr{D}_d f). \tag{10}$$

Also, let

$$d\lambda \equiv \left(dx; \tfrac{1}{2}\, d\sigma_1, \ldots, \tfrac{1}{2}\, d\sigma_d\right). \tag{11}$$

For a subset $E$ of $\mathbb{R}^d$, let $\mathscr{B}_E$ be the Borel $\sigma$-field of $E$ and $\mathscr{B}(E)$ denote the set of functions which are $\mathscr{B}_E$-measurable. Let

$$\begin{aligned} L^j(S, d\lambda) \equiv \Big\{ & g = (g_0; g_1, \ldots, g_d) \in \mathscr{B}(S) \times \mathscr{B}(F_1) \times \cdots \times \mathscr{B}(F_d): \\ & \int_S |g_0|^j\, dx + \tfrac{1}{2} \sum_{i=1}^{d} \int_{F_i} |g_i|^j\, d\sigma_i < \infty \Big\}, \qquad j = 1, 2, \ldots, \end{aligned} \tag{12}$$

and for $g \in L^1(S, d\lambda)$, let

$$\int_S g\,d\lambda \equiv \int_S g_0\,dx + \tfrac{1}{2}\sum_{i=1}^{d}\int_{F_i} g_i\,d\sigma_i.$$

For $g, h \in \mathscr{B}(S) \times \mathscr{B}(F_1) \times \cdots \times \mathscr{B}(F_d)$ we put $g \cdot h \equiv (g_0 h_0; g_1 h_1, \dots, g_d h_d)$, and for $h > 0$, we put $g/h \equiv (g_0/h_0; g_1/h_1, \dots, g_d/h_d)$. With this notation, the basic adjoint relationship (7) can be rewritten as

$$\int_S (\mathscr{A}f \cdot p)\,d\lambda = 0 \quad \text{for all } f \in C_b^2(S). \tag{13}$$

**4. An algorithm.** In this section we develop an algorithm to determine the stationary density $p$ (interior density $p_0$ and boundary densities $p_i$) of an SRBM. We start with the inner product version (13) of our basic adjoint relationship. If $\mathscr{A}f$ and $p$ were in $L^2(S, d\lambda)$ for all $f \in C_b^2(S)$, then (13) would amount to saying that $p$ is orthogonal to $\mathscr{A}f$ for each $f \in C_b^2(S)$. Unfortunately, there are $f \in C_b^2(S)$ for which $\mathscr{A}f$ is not in $L^2(S, d\lambda)$, since the state space $S$ is unbounded. Nevertheless, the above observation is the key to the algorithm that we are going to describe. In fact, based on the above observation, Dai and Harrison [3] developed an algorithm for approximating the stationary density of an SRBM in a two-dimensional rectangular state space.

In order to carry over the algorithm in [3] to the case with unbounded state space, we need to introduce the notion of a reference measure. Let

$$q = (q_0; q_1, \dots, q_d),$$

where $q_0$ is a strictly positive probability density in $S$ and $q_i$ is a strictly positive integrable function on $F_i$ [with respect to the $(d-1)$-dimensional Lebesgue measure $d\sigma_i$]. The function $q$ will be called a reference density. We will come back to the question of how to choose a reference density in the next section. Given a reference density $q$, we define the reference measure

$$d\eta \equiv q\,d\lambda = \left(q_0\,dx; \tfrac{1}{2}q_1\,d\sigma_1, \dots, \tfrac{1}{2}q_d\,d\sigma_d\right), \tag{14}$$

where the measure $d\lambda$ is defined in (11). Similar to the definition of $L^i(S, d\lambda)$ and $\int_S g\,d\lambda$ for $g \in L^1(S, d\lambda)$, we can define $L^i(S, d\eta)$ and $\int_S g\,d\eta$ for $g \in L^1(S, d\eta)$. If we introduce a new unknown $r = p/q$, then the basic adjoint relationship (13) takes the form

$$\int_S (\mathscr{A}f \cdot r)\,d\eta = 0 \quad \text{for all } f \in C_b^2(S). \tag{15}$$

In the following, we actually develop an algorithm to solve for this new unknown $r$. Of course, once one has $r$, one can get the stationary density $p$ via $p = r \cdot q$.

We denote by $L^2 \equiv L^2(S, d\eta)$ all the square integrable functions on $S$ with respect to $d\eta$, taken with the usual inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Unless specified otherwise, all inner products and norms are taken in

$L^2(S, d\eta)$. Because $\eta$ is a finite measure and $\mathscr{A}f$ is bounded, we have $\mathscr{A}f \in L^2$ for each $f \in C_b^2(S)$. We define $H \subset L^2$ to be the closed subspace

$$H \equiv \text{the closure of } \{\mathscr{A}f\colon\ f \in C_b^2(S)\}, \tag{16}$$

where the closure is taken in the usual $L^2$ norm. Let $\phi_0$ be the element of $L^2$ defined by

$$\phi_0 = (1; 0, \dots, 0). \tag{17}$$

PROPOSITION 5. *If the unknown function $r \in L^2$, then $r$ is orthogonal to $H$ and $\langle r, \phi_0 \rangle = 1$. Conversely, if there is a nonnegative $r \in L^2$ such that $r$ is orthogonal to $H$ and $\langle r, \phi_0 \rangle = 1$, then $r \cdot q$ is the stationary density, where $q$ is the chosen reference density.*

PROOF. Assume $r \in L^2$. Since $r$ satisfies the basic adjoint relationship (15), $r$ is orthogonal to $\mathscr{A}f$ for every $f \in C_b^2(S)$. It follows that $r$ is orthogonal to $H$. Also,

$$\begin{aligned}\langle r, \phi_0 \rangle &= \int_S r(x) \cdot \phi_0(x)\, d\eta = \int_S \frac{p}{q} \cdot \phi_0 \cdot q\, d\lambda \\ &= \int_S p \cdot \phi_0\, d\lambda = \int_S p_0\, dx = 1.\end{aligned}$$

The last equality holds because $p_0$ is a probability density function on $S$.

Conversely, assume there is a nonnegative $r \in L^2$ such that $r$ is orthogonal to $H$. Then $r$ is orthogonal to $\mathscr{A}f$ for every $f \in C_b^2(S)$, or equivalently, $r$ satisfies the basic adjoint relationship (15). Therefore, by definition, $r \cdot q$ satisfies the basic adjoint relationship (13). Since $r \cdot q$ is nonnegative and $\int_S (r \cdot q)\, dx = \langle r, \phi_0 \rangle = 1$, it follows from Proposition 4 that $r \cdot q$ is the stationary density of the corresponding SRBM. □

PROPOSITION 6. *If $r \in L^2$, then the function $\tilde{\phi}_0 \equiv \phi_0 - \bar{\phi}_0$ is nonzero in $L^2$ and is orthogonal to $H$, where $\bar{\phi}_0$ is the projection of $\phi_0$ onto $H$ defined by*

$$\bar{\phi} \equiv \operatorname*{arg\,min}_{\phi \in H} \|\phi_0 - \phi\|. \tag{18}$$

*Therefore, assuming Conjecture 1 to be true, we arrive at the following formula for the stationary density:*

$$p = \frac{1}{\|\tilde{\phi}_0\|^2} \tilde{\phi}_0 \cdot q. \tag{19}$$

PROOF. Assume $r \in L^2$. From Proposition 5, $\langle r, \phi_0 \rangle = 1$, and thus $r$ is *not* orthogonal to $\phi_0$. Also from Proposition 5, we know that $r$ is orthogonal to $H$. Hence we conclude that $\phi_0$ is *not* in $H$. Because $H$ is closed and $\phi_0 \notin H$, the projection $\bar{\phi}_0$ defined in (18) of $\phi_0$ is not equal to $\phi_0$, which implies that $\tilde{\phi}_0$ is not zero in $L^2$. Of course, $\tilde{\phi}_0$ is orthogonal to $H$. Note that

$$\langle \tilde{\phi}_0, \phi_0 \rangle = \langle \phi_0 - \bar{\phi}_0, \phi_0 \rangle = \langle \phi_0 - \bar{\phi}_0, \phi_0 - \bar{\phi}_0 \rangle$$
$$= \|\tilde{\phi}_0\|^2 > 0.$$

If Conjecture 1 is true, then $\tilde{\phi}_0$ does not change sign and $r \equiv \langle \tilde{\phi}_0, \phi_0 \rangle^{-1} \tilde{\phi}_0$ satisfies all the conditions in Proposition 5. Then (19) follows from Proposition 5. □

PROPOSITION 7. *Let $\{H_n\}$ be a sequence of finite-dimensional subspaces of $H$, such that $H_n \uparrow H$. ($H_n \uparrow H$ means that $H_1, H_2, \cdots$ are increasing and every $h \in H$ can be approximated by a sequence $\{h_n\}$ with $h_n \in H_n$ for each $n$.) For each $n$, define $\phi^n$ by $\phi^n \equiv \phi_0 - \psi^n$, where*

$$\psi^n \equiv \arg\min_{\phi \in H_n} \|\phi_0 - \phi\|. \tag{20}$$

*If $r \in L^2$, and if we assume Conjecture 1 to be true, then*

$$r^n \equiv \frac{\phi^n}{\|\phi^n\|^2} \to r \quad \text{in } L^2 \text{ as } n \to \infty. \tag{21}$$

*Furthermore, by setting $p^n = r^n \cdot q$, one has for all $f \in L^2$,*

$$\int_S f \cdot p^n \, d\lambda \to \int_S f \cdot p \, d\lambda,$$

*and if $q$ is taken to be bounded, then $p^n \to p$ in $L^2(S, d\lambda)$ as $n \to \infty$.*

PROOF. Since $H_n \uparrow H$, $\psi^n \to \bar{\phi}_0$ in $L^2$ as $n \to \infty$. It follows that $\phi^n \to \tilde{\phi}_0$ in $L^2$ as $n \to \infty$. Because $r \in L^2$, $\tilde{\phi}_0$ is nonzero in $L^2$. Therefore $\|\phi^n\|^2 \to \|\tilde{\phi}_0\|^2 \neq 0$ and hence $r^n \equiv \phi^n / \|\phi^n\|^2$ goes (in $L^2$) to $\tilde{\phi}_0 / \|\tilde{\phi}_0\|^2$, which is $r$ under the assumption that Conjecture 1 is true. If $f \in L^2$, then

$$\begin{aligned} \left| \int_S f \cdot p^n \, d\lambda - \int_S f \cdot p \, d\lambda \right| &= \left| \int_S f (r^n - r) q \, d\lambda \right| \\ &= \left| \int_S f (r^n - r) \, d\eta \right| \le \|f\|^{1/2} \|r^n - r\|^{1/2}. \end{aligned} \tag{22}$$

If $q$ is bounded, then

$$\int_S |p^n - p|^2 \, d\lambda = \int_S |r^n - r|^2 q^2 \, d\lambda \le \left[ \max_{x \in S} q(x) \right]^2 \|r^n - r\|^2. \tag{23}$$

The rest of the proof can be readily obtained from (22) and (23). □

Proposition 7 says that, when $r \in L^2$, we can calculate the corresponding stationary density $p$ numerically by choosing appropriate finite-dimensional subspaces $H_n$. However, when $r \notin L^2$, we can still define $p^n$ via $r^n$ as in (21).

We conjecture that, in this case, $p^n$ converges to $p$ weakly, that is,

$$\int_S f\cdot p^n\,d\lambda \to \int_S f\cdot p\,d\lambda \quad \text{as } n\to\infty \text{ for all } f\in C_b(S).$$

**5. Choosing a reference density and $\{H_n\}$.** In this section we will choose a particular $q$ to serve as the reference density function of the previous section. We will first define some quantities that are of interest in the queueing theoretic application of SRBM. Let $p = (p_0; p_1, \dots, p_d)$ be the stationary density of an SRBM $Z$, and for $i = 1, 2, \dots, d$ let

$$m_i \equiv \int_S (x_i \cdot p_0(x))\,dx. \tag{24}$$

Then $m \equiv (m_1, \dots, m_d)'$ represents the long-run mean position of the SRBM $Z$. In the applications mentioned earlier, $m_i$ approximates a performance measure of the corresponding queueing network (for example, the average waiting time or average queue length at station $i$). There are, of course, other quantities associated with $p_0$ that are of interest, such as the second moments or quantities like $\int_S \max(x_i, x_j)p_0(x)\,dx$ for $i \neq j$; see Nguyen [17]. Because our algorithm gives estimates for the density function itself, such extensions are routine and we will only focus on the quantity $m$. Proposition 8 indicates that, in calculating the steady-state mean vector $m$, it is enough to consider only standard SRBM's.

DEFINITION 5. A semimartingale RBM with data $(S, \Gamma, \mu, R)$ is said to be standard if $\operatorname{diag}(\Gamma) = \operatorname{diag}(R) = I$, $R$ is invertible and $\max_{1\le i\le d}|(R^{-1}\mu)_i| = 1$.

PROPOSITION 8. *Suppose that $Z$ with $\{P_x, x\in S\}$ is an SRBM with data $(S, \Gamma, \mu, R)$, that $Z$ has a stationary distribution (thus $R$ is invertible by Proposition 2), and that the steady-state mean vector $m = (m_1, \dots, m_d)'$ is defined via (24). Let $\gamma = -R^{-1}\mu$, $\Lambda = \operatorname{diag}(\Gamma_{11}^{-1/2}, \dots, \Gamma_{dd}^{-1/2})$ and $a = \max_{1\le i\le d}(\Gamma_{ii}^{-1/2}R_{ii}\gamma_i)$. The process $Z^* = \{Z^*(t), t\ge 0\}$ defined by*

$$Z^*(t) = a\Lambda Z(a^{-2}t), \qquad t\ge 0,$$

*is a standard SRBM with data $(S, \Gamma^*, \mu^*, R^*)$, where*

$$\Gamma^* = \Lambda\Gamma\Lambda, \qquad \mu^* = a^{-1}\Lambda\mu, \qquad R^* = \Lambda R V\Lambda^{-1}$$

*and $V = \operatorname{diag}(R_{11}^{-1}, \dots, R_{dd}^{-1})$. Moreover, $Z^*$ has a stationary distribution and its associated mean vector $m^*$ is related to $m$ via*

$$m^* = a\Lambda m.$$

PROOF. Suppose that $Z$ with $\{P_x, x\in S\}$ is a $(S, \Gamma, \mu, R)$ SRBM. We know that $Z$ can be represented in the form

$$Z(t) = X(t) + RL(t), \qquad t\ge 0,$$

where $X$ is a $(\Gamma, \mu)$ Brownian motion and $L$ is an increasing process that has the properties specified in Definition 1. Thus we can represent $Z^*$ in the form

$$Z^*(t) = X^*(t) + R^*L^*(t), \qquad t \geq 0,$$

where $X^*(t) = a\Lambda X(a^{-2}t)$ and $L^*(t) = a\Lambda V^{-1}L(a^{-2}t)$. [Note that $R^*$ and $L^*$ are defined so as to ensure $\operatorname{diag}(R^*) = I$.] It is easy to verify that $X^*$ is a $(\Gamma^*, \mu^*)$ Brownian motion and that $L^*$ and $X^*$ jointly satisfy the obvious analogs of (1) and (3). Thus $Z^*$ is an SRBM with data $(S, \Gamma^*, \mu^*, R^*)$. Moreover, defining $\gamma^* \equiv -R^{*-1}\mu^*$ in the obvious way, one has that

$$\begin{aligned}\gamma^* &= -(\Lambda V^{-1}R^{-1}\Lambda^{-1})(a^{-1}\Lambda\mu) = -a^{-1}\Lambda V^{-1}R^{-1}\mu \\ &= a^{-1}\Lambda V^{-1}\gamma.\end{aligned}$$

Thus

$$\max_{1\leq i\leq d} |\gamma_i^*| = a^{-1} \max_{1\leq i\leq d} |\Gamma_{ii}^{-1/2}R_{ii}\gamma_i| = 1$$

by the definition of $a$, and $\Lambda$ has been chosen so as to give $\operatorname{diag}(\Gamma^*) = I$, so $Z^*$ is a *standard* SRBM as claimed. Finally, let $Z(\infty)$ be a random vector whose distribution is the stationary distribution of $Z$. Then $Z^*(\infty) \equiv a\Lambda Z(\infty)$ has the stationary distribution of $Z^*$, implying that

$$m^* \equiv E[Z^*(\infty)] = a\Lambda E[Z(\infty)] \equiv a\Lambda m. \qquad \square$$

When the SRBM is standard, all the associated data are properly scaled and therefore the algorithm described in the previous section is more stable.

DEFINITION 6. We say that a stationary density for $Z$ is of product form (or has a separable density) if the stationary density $p_0$ can be written as

$$(25) \qquad p_0(z) = \prod_{k=1}^{d} p_0^k(z_k), \qquad z = (z_1, \ldots, z_k) \in S,$$

where $p_0^1, \ldots, p_0^d$ are all probability densities relative to Lebesgue measure on $\mathbb{R}_+$. The following is proved by Harrison and Williams ([13], Theorem 9.2); in that paper it is assumed that the reflection matrix $R$ is a *Minkowski matrix*, but that additional assumption does not enter in the proof of this proposition.

PROPOSITION 9. *A standard SRBM Z has a product form stationary distribution if and only if*

$$(26) \qquad \gamma \equiv -R^{-1}\mu > 0$$

*holds, as well as the following condition*:

$$(27) \qquad 2\Gamma_{jk} = (R_{kj} + R_{jk}) \qquad \text{for } j \neq k.$$

*In this case, there is a constant C such that the density p is the exponential*

$$(28) \qquad z \to C\exp(-2\gamma \cdot z), \qquad z = (z_1, \ldots, z_d) \in S,$$

*that is, $p_0$ is given by* (28) *and $p_i$ is the restriction of this exponential to $F_i$ ($i = 1, 2, \ldots, d$), where $\gamma$ is defined in* (26).

REMARK. Condition (27) holds if and only if $\Gamma - R$ is skew symmetric; Harrison and Williams [13] refer to (27) as a *skew symmetry* condition.

Proposition 9 asserts that the density $p$ is of exponential form precisely when the skew symmetry condition (27) is satisfied. If we choose $q$ to be the exponential in (28), then $r \equiv p/q$ is identically 1 when (27) is satisfied. When the skew symmetry condition (27) is not satisfied, but is almost satisfied, we expect the density $p$ to be only slightly perturbed from the above exponential. That is, $r$ is nearly equal to 1. Therefore, we can think of $r$ as some adjusting factor of how far the actual stationary density $p$ is from the exponential solution. Based on these observations, we choose $q$ to be the exponential in (28).

COROLLARY 1. *Fix the reference density $q$ to be the exponential in* (28) *and let $d\eta$ be as in* (14). *If $r \equiv p/q$ is in $L^2(S, d\eta)$, then for each $f \in L^2(S, d\eta)$,*

$$\int_S f(x)p^n(x)\,dx \to \int_S f(x)p(x)\,dx \quad \text{as } n \to \infty.$$

*In particular, the approximating moment $m_i^n \equiv \int_S x_i p^n\,dx$ converges to $m_i$ $(i = 1, 2, \ldots, d)$.*

PROOF. Since $q$ is decaying exponentially, $f(x) \equiv x_i \in L^2 = L^2(S, d\eta)$. Therefore, the proof of the corollary is an immediate consequence of Proposition 7.

The next proposition tells us how to choose a finite-dimensional subspace $H_n$ approximating $H$. Here we assume that the reference density $q$ has been chosen as in Corollary 1.

PROPOSITION 10. *For each integer $n \geq 1$, let*

$$H_n = \{\mathscr{A}f\colon f \text{ is a polynomial of degree } \leq n\}. \tag{29}$$

*Then $H_n \uparrow H$.*

PROOF. It is obvious that $H_n$ is increasing. The proof of the convergence $H_n \to H$ is an immediate consequence of Proposition 7.1 and Remark 6.2 in the appendices of Ethier and Kurtz [6]. □

REMARK. Since the chosen $q$ is smooth, our approximating function $p^n = r^n \cdot q$ is also smooth, whereas in many cases the stationary density $p$ is known to be singular. In some cases, these singularities will cause $r \equiv p/q \notin L^2$. If we knew in advance the order of the singularities at the nonsmooth parts of the boundary, we might be able to incorporate these singularities into the reference density $q$, so that $r = p/q$ would be smooth and in $L^2$. This would yield an algorithm which would converge faster. Unfortunately, there is no general result on the order of the singularities of the stationary density of an SRBM in high dimensions.

**6. Numerical comparisons.** We have written a computer program in "$C$" that implements the general algorithm described in Section 4. In this initial implementation the reference density $q$ is the exponential function specified in (28) and the finite-dimensional subspaces $H_n$ are those specified in (29). As a basis for $H_n$, we take the following collection of functions on $\mathbb{R}^d$:

$$B_n \equiv \{\mathscr{A}(x_1^{i_1} \cdots x_d^{i_d}) : 0 < i_1 + \cdots + i_d \le n \text{ and } i_1, \ldots, i_d \text{ are nonnegative integers}\}.$$

Let $d$ and $n$ be fixed for the moment and define

$$N = \binom{n+d}{d} = (d+1)\cdots(d+n)/n!.$$

The basis $B_n$ contains $N-1$ elements and to determine our estimate $p^n$ of the steady-state density $p$, one must compute the projection $\psi^n$ of $\phi^0$ onto the span of $B_n$. (The time required for the remainder of the algorithm is trivial and will not even be mentioned hereafter.) One familiar and elementary approach to computing the projection $\psi^n$ requires that one first solve a certain system of linear equations $Kx = b$. To calculate the coefficient matrix $K$ and the vector $b$, one must evaluate inner products for pairs of functions in $B_n$, which requires a general formula for the integral of a monomial times an exponential function over the nonnegative orthant. Given that formula, the number of calculations required to compute $\psi^n$ and hence $p^n$ is $O(N^2)$.

With the implementation described in the last paragraph, the estimate $p^n$ that we ultimately obtain for the steady-state density $p$ is an $(n-1)$th degree polynomial times the exponential reference density $q$. One expects that larger values of $n$ will give better accuracy, and as a practical matter we have found that $n = 5$ generally gives satisfactory answers, at least for the test problems examined thus far. If one fixes $n = 5$, the computational complexity of the algorithm is $O(d^{10})$, which means that small- and medium-sized problems can be solved using the current implementation of the algorithm. To give readers a feel for actual running times, we have examined a family of seven closely related SRBM's having dimension $2, 3, \ldots, 8$. The member of this family having dimension $d$ corresponds to the first $d$ stations of a particular eight-station tandem queue, the exact data of which are not relevant for current purposes. The amount of CPU time (in seconds) on a SUN SPARCstation 1 required to compute $p^n$ for each case was as follows:

| | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ | $d = 8$ |
|---|---|---|---|---|---|---|---|
| $n = 4$ | 0.0 | 0.2 | 1.2 | 6.0 | 23.6 | 80.5 | 251.1 |
| $n = 5$ | 0.0 | 0.6 | 4.9 | 31.9 | 162.9 | 734.2 | 2654.8 |

Some early experience regarding the accuracy of the algorithm will be presented in this section. The Ph.D. dissertation of Dai [2] describes more computational experiences and explains in detail the current implementation. A future paper by Dai, Nguyen and Reiman [5] will describe a decomposition method that can be used on large and very large queueing networks.

In the remainder of this section we will compare numerical results (called QNET estimates) obtained with the current implementation of our algorithm against some known analytical results for particular SRBM's. The two cases discussed in this section are the only cases for which explicit analytical solutions are known, except for SRBM's with exponential stationary densities. Because the exponential solutions are incorporated in our algorithm, the cases discussed below represent the only exact solutions available for checking the algorithm.

*A two-dimensional* SRBM. Consider a two-dimensional SRBM with covariance matrix $\Gamma = I$, drift vector of the form $\mu = (\mu_1, 0)'$ and reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

An SRBM with these data $(\Gamma, \mu, R)$ arises as a Brownian approximation for a special type of tandem queue. For this SRBM, our stationary condition (26) reduces to $\mu_1 < 0$. This type of SRBM was studied in Harrison [11]. There, after a transformation, the author was able to obtain a solution of product form for the stationary density in polar coordinates. The explicit form of the stationary density is

$$(30) \quad p(x) = Cr^{-1/2}e^{\mu_1(r+x_1)}\cos(\theta/2), \qquad x = (x_1, x_2) = (r\cos\theta, r\sin\theta),$$

where $C = \pi^{-1/2}(2|\mu_1|)^{3/2}$. (Greenberg [10] pointed out that Harrison's [11] original calculation of $C$ was in error by a factor of 2.) Notice that $x = 0$ is the singular point of the density. The above density $p$ is square integrable in $S$ with respect to interior Lebesgue measure, but $p$ is *not* square integrable over the boundary with respect to boundary Lebesgue measure. Therefore, $p$ is not in $L^2$. By the scaling argument given in Proposition 8, it is enough to consider the case when $\mu_1 = -1$. It follows from (30) (cf. Greenberg [10]) that

$$m_1 = \tfrac{1}{2}, \qquad m_2 = \tfrac{3}{4}.$$

Using our algorithm, taking the maximum degree of the polynomials in (29) to be $n = 5$, we have QNET estimates

$$m_1 = 0.50000, \qquad m_2 = 0.75133.$$

The QNET estimate of $m_1$ is exact as expected. If one takes the first station in the tandem queue in isolation, the first station will correspond to a one-dimensional SRBM, whose stationary density is always of exponential form. It was rigorously proved in [11] that the one-dimensional marginal distribution of $x_1$ is indeed of exponential form. Table 1 shows that if we require 1% accuracy, which is usually good enough in queueing network applications, the conver-

TABLE 1
*Convergence of the algorithm*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $m_2$ | 0.50000 | 0.83333 | 0.75000 | 0.75873 | 0.75133 | 0.75334 | 0.75225 | 0.75681 |

gence is very fast, even for this very singular density. It appears that the accuracy of $m_2$ does not increase as $n$ increases. This shows that for very singular $r$, the numerical roundoff errors in the approximation may have a significant effect on the accuracy of the algorithm.

*Symmetric SRBM's.* A standard SRBM (cf. Definition 5) is said to be *symmetric* if its data $(\Gamma, \mu, R)$ are symmetric in the following sense: $\Gamma_{ji} = \Gamma_{ij} = \rho$ for $1 \le i < j \le d$, $\mu_i = -1$ for $1 \le i \le d$ and $R_{ij} = R_{ji} = -r$ for $1 \le i \le j \le d$, where $r \ge 0$. The positiveness of $\Gamma$ implies $-1/(d-1) < \rho < 1$ and the completely-$\mathscr{S}$ condition of $R$ implies $r(d-1) < 1$. A symmetric SRBM arises as a Brownian approximation of a symmetric *generalized Jackson network*. In such a network, each of the $d$ stations behaves exactly the same. Customers finishing service at one station will go to any one of the other $d-1$ stations with equal probability $r$ and will leave the network with probability $1-(d-1)r$. For $d = 2$, the symmetric queueing network was used by Foschini to model a pair of communicating computers [8]. The author extensively studied the stationary density of the corresponding two-dimensional symmetric SRBM.

Because the data $(\Gamma, \mu, R)$ of a symmetric SRBM is, in an obvious sense, invariant under permutation of the integer set $\{1, 2, \dots, d\}$, it is clear that the stationary density $p_0(x)$ is symmetric, that is,

$$p_0(x_1, x_2, \dots, x_d) = p_0(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)})$$

for any permutation $\sigma$ on $\{1, 2, \dots, d\}$. In particular, $\int_{F_j} x_i \cdot p_j \, d\sigma_j = \delta \equiv \int_{F_1} x_2 \cdot p_1 \, d\sigma_1$ for all $i \ne j$ and the marginal densities of $p_0$ are the same, and hence

$$m_1 = m_2 = \cdots = m_d.$$

If we take $f = x_1^2$, then the basic adjoint relationship (7) gives

$$1 - 2m_1 - r \sum_{j=2}^{d} \int_{F_j} x_1 \cdot p_j \, d\sigma_j = 0. \tag{31}$$

Taking $f = x_1 x_2$, we have

$$\begin{aligned} &\rho - (m_1 + m_2) + \tfrac{1}{2} \int_{F_1} x_2 \cdot p_1 \, d\sigma_1 + \tfrac{1}{2} \int_{F_2} x_1 \cdot p_2 \, d\sigma_2 \\ &\quad - \tfrac{1}{2} r \sum_{j=3}^{d} \int_{F_j} (x_1 + x_2) p_j \, d\sigma_j = 0. \end{aligned} \tag{32}$$

By symmetry, from (31) and (32) we get

$$1 - 2m_1 - \delta(d-1)r = 0, \tag{33}$$

$$\rho - 2m_1 + \delta - \delta(d-2)r = 0. \tag{34}$$

TABLE 2
QNET *estimates for* $m_1$ *when* $d = 2$ $(n = 3)$

| $r \setminus \rho$ | −0.90 | −0.50 | 0.00 | 0.50 | 0.90 |
|---|---|---|---|---|---|
| 0.20 | 0.341667 | 0.375000 | 0.416667 | 0.458333 | 0.491667 |
| 0.40 | 0.228571 | 0.285714 | 0.357143 | 0.428571 | 0.485714 |
| 0.60 | 0.143750 | 0.218750 | 0.312500 | 0.406250 | 0.481250 |
| 0.80 | 0.077778 | 0.166667 | 0.277778 | 0.388889 | 0.477778 |
| 0.90 | 0.050000 | 0.144737 | 0.263158 | 0.381579 | 0.476316 |
| 0.95 | 0.037179 | 0.134615 | 0.256410 | 0.378205 | 0.475641 |

TABLE 3
*Relative errors when* $d = 2$ $(n = 3)$

| $r \setminus \rho$ | −0.90 | −0.50 | 0.00 | 0.50 | 0.90 |
|---|---|---|---|---|---|
| 0.20 | 9.75*e*−16 | 4.44*e*−16 | −7.99*e*−16 | −2.42*e*−16 | 3.39*e*−16 |
| 0.40 | 3.89*e*−15 | −7.77*e*−16 | −7.77*e*−16 | 5.18*e*−16 | 1.49*e*−15 |
| 0.60 | 4.63*e*−15 | 3.17*e*−15 | 8.88*e*−16 | 2.73*e*−15 | −2.42*e*−15 |
| 0.80 | 2.32*e*−15 | −4.33*e*−15 | −1.20*e*−15 | −2.63*e*−14 | 3.83*e*−15 |
| 0.90 | −6.91*e*−14 | 2.45*e*−14 | −2.08*e*−13 | −1.81*e*−12 | −7.46*e*−13 |
| 0.95 | −9.59*e*−14 | 6.69*e*−13 | 5.44*e*−12 | 3.32*e*−11 | 1.04*e*−10 |

Solving these linear equations gives $\delta = (1 - \rho)/(1 + r)$ and

$$m_1 = \frac{1 - (d-2)r + (d-1)r\rho}{2(1+r)}. \tag{35}$$

Now we compare our numerical estimates of $m_1$ with the exact values of $m_1$ calculated from formula (35). When $d = 2$, the conditions on the data $(\Gamma, \mu, R)$ yield $|\rho| < 1$ and $0 \le r < 1$. Letting $\rho$ range through $\{-0.9, -0.5, 0.0, 0.5, 0.9\}$ and $r$ range through $\{0.2, 0.4, 0.6, 0.8, 0.9, 0.95\}$ and taking $n = 3$, we obtain QNET estimates for $m_1$ (Table 2). Table 3 gives the relative errors between these QNET estimates and the exact values.

When $r = 1$, there is no corresponding SRBM. It is expected that when $\rho$ is big [the skew symmetry condition (27) is far from being satisfied], the stationary density is very singular as $r \uparrow 1$. This phenomenon seems to be indicated in Table 3, where the performance of the algorithm degrades as $r$ increases to 1. When the dimension $d$ is 3, then the restriction on the data gives $-\frac{1}{2} < \rho < 1$ and $0 \le r < \frac{1}{2}$. Table 4 gives the relative errors between some QNET estimates and the exact values for $m_1$ in this case. When the dimension $d$ is 4, then the restriction on the data gives $-\frac{1}{3} < \rho < 1$ and $0 \le r < \frac{1}{3}$; relative errors between QNET estimates and exact values for $m_1$ are found in Table 5.

**7. Analysis of an illustrative queueing network.** The two-station open queueing network pictured in Figure 2 has been suggested by Gelenbe

TABLE 4
*Relative errors when $d = 3$ ($n = 3$)*

| $r \setminus \rho$ | **−0.40** | **−0.20** | **0.00** | **0.80** | **0.90** |
|---|---|---|---|---|---|
| 0.10 | 1.49e−16 | 1.14e−15 | −5.43e−16 | −4.61e−16 | −1.47e−15 |
| 0.30 | 2.04e−15 | −2.12e−15 | −2.68e−15 | −9.79e−15 | −1.75e−14 |
| 0.40 | 1.39e−15 | −1.06e−15 | −7.77e−15 | 8.65e−14 | 3.66e−13 |
| 0.45 | −1.06e−14 | −6.16e−14 | −6.54e−14 | 9.21e−12 | 7.73e−12 |

TABLE 5
*Relative errors when $d = 4$ ($n = 3$)*

| $r \setminus \rho$ | **−0.30** | **−0.15** | **−0.00** | **0.80** | **0.90** |
|---|---|---|---|---|---|
| 0.10 | −2.06e−15 | −1.29e−15 | 9.16e−16 | −3.88e−15 | −5.14e−15 |
| 0.20 | 4.60e−15 | 7.18e−15 | 8.88e−16 | 7.40e−16 | 2.34e−14 |
| 0.30 | −2.01e−14 | 1.39e−14 | −1.08e−14 | 3.23e−12 | 4.78e−12 |
| 0.33 | −2.23e−13 | −5.32e−12 | −6.06e−12 | 4.60e−10 | 1.53e−11 |

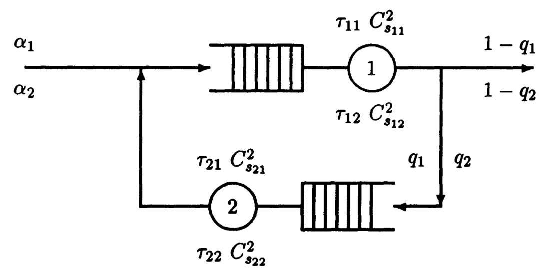


FIG. 2. *Model of an interactive computer.*

and Pujolle [9] as a simplified model of a certain computer system. Server 1 represents a central processing unit (CPU) and server 2 a secondary memory. There are two classes of programs (jobs, or customers) flowing through the system, and they differ in their relative use of the CPU and the secondary memory. Jobs of class $j$ ($j = 1, 2$) arrive at station 1 according to a Poisson process with rate $\alpha_j$; and after completing service there they may either go on to station 2 (probability $q_j$) or leave the system (probability $1 - q_j$); each service at station 2 is followed by another service at station 1, after which the customer either returns to station 2 or else leaves the system, again with probability $q_j$ and $1 - q_j$, respectively. The service time distribution for class $j$ customers at station $i$ ($i, j = 1, 2$) is the same on every visit; its mean is $\tau_{ij}$

and its coefficient of variation (standard deviation divided by mean) is $C_{s_{ij}}$. Customers are served on a first-in-first-out basis, without regard to class, at each station. The specific numerical values that we will consider are such that class 1 makes heavier demands on the secondary memory but class 2 consumes more CPU time. Denoting by $Q_i$ ($i = 1, 2$) the long-run average queue length at station $i$, including the customer being served there (if any), our goal is to estimate $Q_1$ and $Q_2$.

This open queueing network is within the class for which Harrison and Nguyen [12] have proposed an approximate Brownian model, but their initial focus is on the *current workload process* or *virtual waiting time process* $W(t) = (W_1(t), W_2(t))'$, rather than the queue length process; one may think of $W_i(t)$ as the time that a new arrival to station $i$ at time $t$ would have to wait before gaining access to the server. Harrison and Nguyen proposed that the process $W(t)$ be modeled or approximated by an SRBM in the quadrant whose data $(\Gamma, \mu, R)$ are derived from the parameters of the queueing system by certain formulas. Specializing those formulas to the case at hand one obtains

$$\mu = R(\rho - e), \qquad \Gamma = TGT' \quad \text{and} \quad R = M^{-1}, \tag{36}$$

where $e$ is the two-vector of ones, $\rho = (\rho_1, \rho_2)'$ is the vector of traffic intensities

$$\rho_1 = \frac{\alpha_1}{1-q_1}\tau_{11} + \frac{\alpha_2}{1-q_2}\tau_{12} \quad \text{and} \quad \rho_2 = \frac{\alpha_1 q_1}{1-q_1}\tau_{21} + \frac{\alpha_2 q_2}{1-q_2}\tau_{22},$$

and the matrices $M$, $T$ and $G$ are given by

$$M = \begin{pmatrix} \dfrac{1}{\rho_1}F_{11} & \dfrac{1}{\rho_2}F_{12} \\ \dfrac{1}{\rho_1}F_{21} & \dfrac{1}{\rho_2}F_{22} \end{pmatrix},$$

where

$$F = \begin{pmatrix} \dfrac{\alpha_1\tau_{11}}{(1-q_1)^2} + \dfrac{\alpha_2\tau_{12}}{(1-q_2)^2} & \dfrac{\alpha_1\tau_{11}q_1}{(1-q_1)^2} + \dfrac{\alpha_2\tau_{12}q_2}{(1-q_2)^2} \\ \dfrac{\alpha_1\tau_{21}q_1}{(1-q_1)^2} + \dfrac{\alpha_2\tau_{22}q_2}{(1-q_2)^2} & \dfrac{\alpha_1\tau_{21}q_1}{(1-q_1)^2} + \dfrac{\alpha_2\tau_{22}q_2}{(1-q_2)^2} \end{pmatrix},$$

$$T = \begin{pmatrix} \dfrac{\tau_{11}}{1-q_1} & \dfrac{\tau_{11}}{1-q_1} & \dfrac{\tau_{12}}{1-q_2} & \dfrac{\tau_{12}}{1-q_2} \\ \dfrac{\tau_{21}q_1}{1-q_1} & \dfrac{\tau_{21}}{1-q_1} & \dfrac{\tau_{22}q_2}{1-q_2} & \dfrac{\tau_{22}}{1-q_2} \end{pmatrix},$$

and

$$G = \begin{pmatrix} \alpha_1 + \dfrac{\alpha_1}{1-q_1} g_1 & -\dfrac{\alpha_1 q_1}{1-q_1} g_{12} & 0 & 0 \\ -\dfrac{\alpha_1 q_1}{1-q_1} g_{12} & \alpha_1 q_1 + \dfrac{\alpha_1 q_1}{1-q_1} g_2 & 0 & 0 \\ 0 & 0 & \alpha_2 + \dfrac{\alpha_2}{1-q_2} g_3 & -\dfrac{\alpha_1 q_2}{1-q_2} g_{34} \\ 0 & 0 & -\dfrac{\alpha_1 q_2}{1-q_2} g_{34} & \alpha_2 q_2 + \dfrac{\alpha_2 q_2}{1-q_2} g_4 \end{pmatrix},$$

where

$$g_1 = \left(C_{s_{11}}^2 + q_1 C_{s_{21}}^2\right), \quad g_2 = \left(C_{s_{21}}^2 + q_1 C_{s_{11}}^2\right), \quad g_{12} = \left(C_{s_{11}}^2 + C_{s_{21}}^2\right),$$
$$g_3 = \left(C_{s_{12}}^2 + q_2 C_{s_{22}}^2\right), \quad g_4 = \left(C_{s_{22}}^2 + q_2 C_{s_{12}}^2\right), \quad g_{34} = \left(C_{s_{12}}^2 + C_{s_{22}}^2\right).$$

Let us denote by $m = (m_1, m_2)'$ the mean vector of the stationary distribution of the SRBM whose data $(\Gamma, \mu, R)$ are computed via (36). In the approximation scheme of Harrison and Nguyen [12], which they call the QNET method, one approximates by $m_i$ both the long-run average *virtual* waiting time and the long-run average *actual* waiting time at station $i$ $(i = 1, 2)$. By Little's law $(L = \lambda W)$, we then have the following QNET estimates of the average queue length at the two station:

$$Q_1 = \rho_1 + \left(\frac{\alpha_1}{1-q_1} + \frac{\alpha_2}{1-q_2}\right) m_1, \tag{37}$$

$$Q_2 = \rho_2 + \left(\frac{\alpha_1 q_1}{1-q_1} + \frac{\alpha_2 q_2}{1-q_2}\right) m_2. \tag{38}$$

Gelenbe and Pujolle [9] have simulated the performance of this simple queueing network in the five different cases described by Table 6, obtaining the results displayed in Table 7. All of the numerical results in the latter table

TABLE 6
*Parameters for the system*
$\alpha_1 = 0.5, \alpha_2 = 0.25, q_1 = 0.5, q_2 = 0.2$

| | Class 1 | | | | Class 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | Station 1 | | Station 2 | | Station 1 | | Station 2 | |
| Case | Mean | SCV | Mean | SCV | Mean | SCV | Mean | SCV |
| 1 | 0.5 | 1.0 | 0.5 | 2.0 | 1.0 | 0.0 | 1.0 | 1.0 |
| 2 | 0.5 | 0.2 | 0.5 | 2.0 | 1.0 | 0.0 | 1.0 | 1.0 |
| 3 | 0.5 | 1.0 | 1.0 | 1.0 | 0.5 | 1.0 | 1.0 | 1.0 |
| 4 | 0.5 | 3.0 | 0.5 | 2.0 | 0.5 | 0.0 | 1.0 | 1.0 |
| 5 | 0.5 | 3.0 | 0.5 | 1.0 | 0.5 | 0.0 | 1.0 | 0.2 |

TABLE 7
*Mean number of customers for the network represented in Figure 2*

| | Case 1 | | Case 2 | | Case 3 | | Case 4 | | Case 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 0.81 | 0.31 | 0.81 | 0.31 | 0.66 | 0.56 | 0.66 | 0.31 | 0.66 | 0.31 |
| $Q$ | $Q_1$ | $Q_2$ | $Q_1$ | $Q_2$ | $Q_1$ | $Q_2$ | $Q_1$ | $Q_2$ | $Q_1$ | $Q_2$ |
| SIM | 3.62 | 0.49 | 3.15 | 0.45 | 1.92 | 1.24 | 2.35 | 0.51 | 2.37 | 0.50 |
| TD | 4.33 | 0.45 | 4.33 | 0.45 | 1.90 | 1.29 | 1.91 | 0.45 | 1.91 | 0.45 |
| DC | 3.66 | 0.50 | 2.91 | 0.49 | 1.90 | 1.29 | 2.77 | 0.53 | 2.77 | 0.50 |
| QNET | 3.83 | 0.50 | 3.40 | 0.49 | 1.90 | 1.29 | 2.48 | 0.54 | 2.41 | 0.48 |

except the QNET estimates are taken from Table 5.3 of [9]: the row labeled SIM gives simulation results, whereas the row labeled TD gives a time division approximation based on the classical theory of product-form queueing networks and that labeled DC gives a diffusion approximation that is essentially Whitt's [23] QNA scheme for two-moment analysis of system performance via node decomposition. In essence, this last method uses a diffusion approximation to the queue length process of each station individually, after artificially decomposing the network into one-station subnetworks; the QNET method captures more subtle system interactions by considering the joint stationary distribution of an approximating two-dimensional diffusion process, the mean vector $m$ of that stationary distribution being computed by means of the algorithm described earlier in this paper.

As Table 7 indicates, our QNET method gives very good approximations, somewhat better overall than either the TD or DC approximations. The network described by case 3 is in fact a product form network, and for it all these approximation schemes give exact results.

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SCHOOL OF MATHEMATICS
AND INDUSTRIAL / SYSTEMS ENGINEERING
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332

GRADUATE SCHOOL OF BUSINESS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305