The Double Skorohod Map and Real-Time Queues

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- J. M. Harrison & S. R. Pliska (1981) Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Applications* **11**, 215–260.

- J. M. Harrison & S. R. Pliska (1983) A stochastic calculus model of continuous trading: complete markets, *Stochastic Processes and Applications* **15**, 313–316.

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Theorem (First Fundamental Theorem)

There exists a martingale measure if and only if a model admits no arbitrage.

Theorem (Second Fundamental Theorem)

Consider a model that admits no arbitrage. The martingale measure is unique if and only if every derivative security can be replicated by trading in the primary assets.

Context

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- Discrete-time trading and continuous-time trading.

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"We are working dangerously close to the boundaries of our knowledge...." — *J. M. Harrison and S. Pliska*

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- No longer must asset price processes be continuous.

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- Risk-neutral pricing, a consequence of the existence of the martingale measure, can be applied blindly without thinking whether the measure is unique.
  See "Did faulty mathematical models cause the financial fiasco?," *Analytics Magazine*, Spring 2009, available at www.math.cmu.edu/users/shreve.

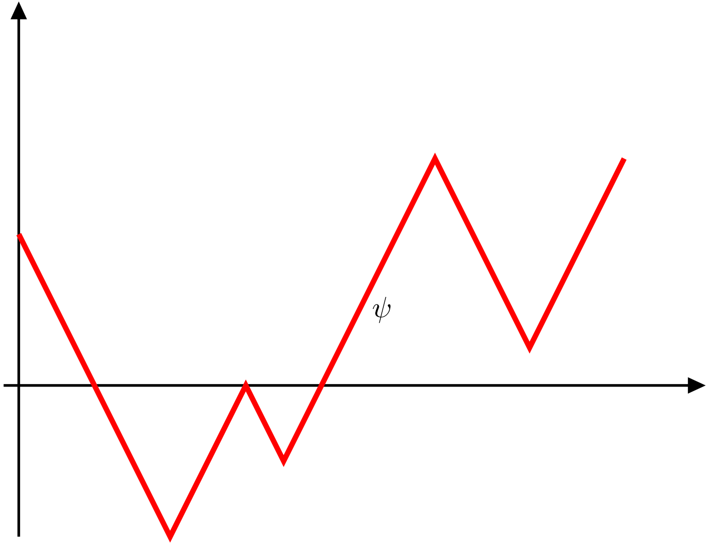
Outline of the Rest of the Talk

Skorohod Map

Real-Time Queues

Real-Time Queues with Reneging

1. Skorohod Map



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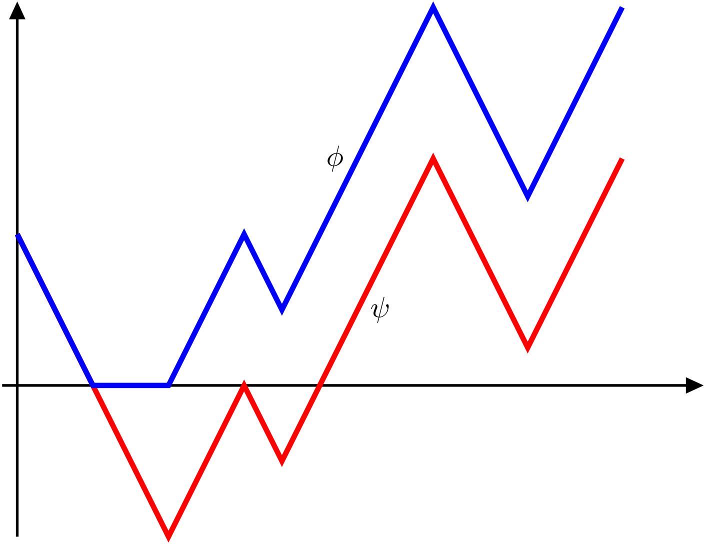

$\phi(t) \triangleq \psi(t) - \inf_{0 \leq s \leq t} \left[\psi(s) \wedge 0\right]$. (Skorokhod, 1961)

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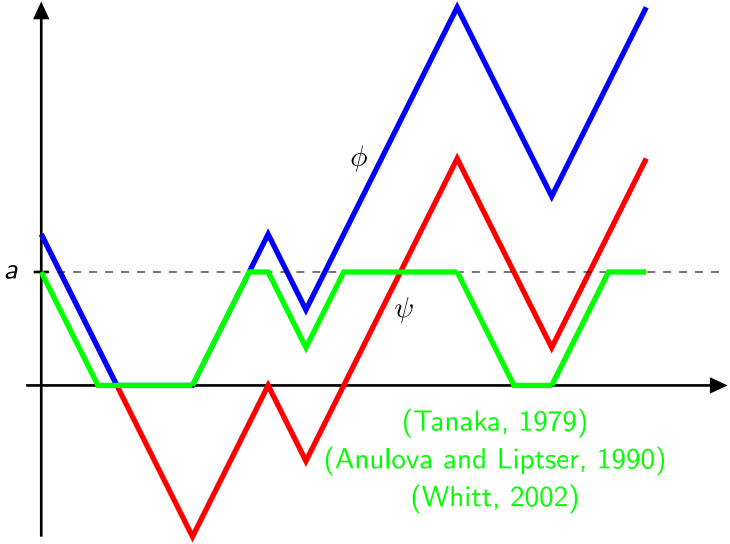
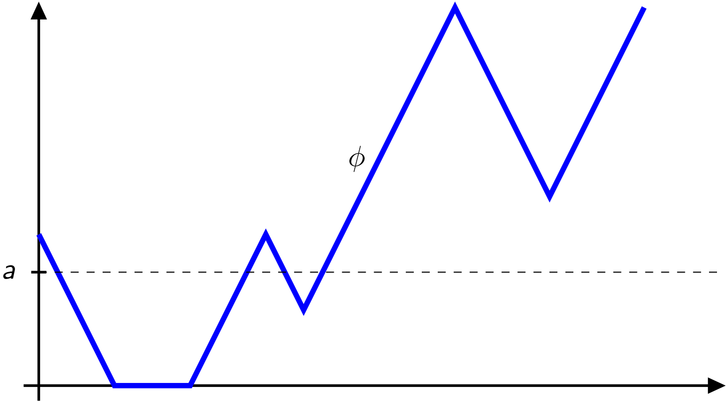


$\phi(t) \triangleq \psi(t) - \inf_{0\leq s\leq t} \left[\psi(s) \wedge 0\right].$ (Skorokhod, 1961)

Formula for Double Skorohod Map



$$\lambda(\phi)(t) \triangleq \phi(t) - \sup_{s\in[0,t]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right]$$

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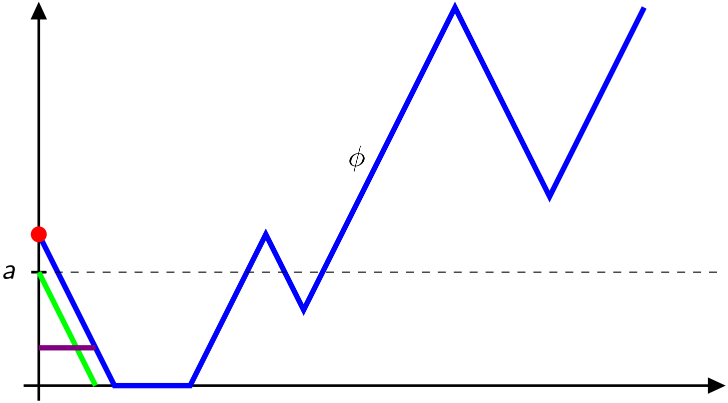

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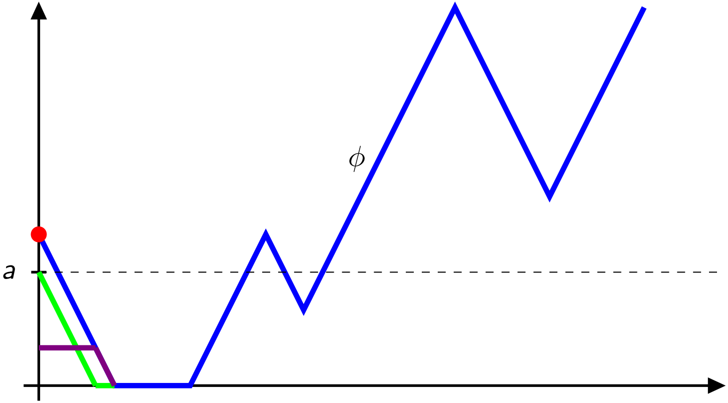

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Related formulas

Toomey (1998).

Let ψ be piecewise constant. The double reflection in $[0,a]$ of ψ is

$$\inf_{s\in(0,t]} \sup_{u\in(s,t]} \Big[\big(a+\psi(t)-\psi(s)\big) \vee \big(\psi(t)-\psi(u)\big)\Big] \vee \sup_{u\in(0,t]} \Big[\big(\psi(t) \vee \big(\psi(t)-\psi(u)\big)\Big].$$

Related formulas

Cooper, Schmidt and Serfozo (2001).
H is a signed measure on $[0, \infty)$ and

$$X(t) = \sup_{s \in [0,t]} \inf_{u \in [s,t]} \left[x I_{\{s=u=o\}} + H(u,t] - a I_{\{s=u>0\}} \right],$$

Then X is the double reflection in $[-a, 0]$ of the bounded-variation function $t \mapsto (x + H(0,t])$.

Related formulas

Ganesh, O'Connell and Wischik (2004).
Let ψ be a bounded-variation function. The double reflection in $[0, a]$ of ψ is

$$\left(\psi(t) \vee \inf_{s\in[0,t]} \left[N(s,t) \wedge \left(M(s,t)+a\right)\right]\right) \wedge \inf_{s\in[0,t]} \left[N(s,t) \vee \left(M(s,t)+a\right)\right],$$

where

$$\begin{aligned} M(s,t) &= \psi(t) - \sup_{u\in[s,t]} \psi(u), \\ N(s,t) &= \psi(t) - \inf_{u\in[s,t]} \psi(u). \end{aligned}$$

2. Real-Time Queues

Single station, renewal arrival process.

Heavy traffic assumption: For some $\gamma \neq 0$, $\rho^{(n)} = 1 - \frac{\gamma}{\sqrt{n}}$.

Workload process: $W^{(n)}(t)$

Scaled workload process: $\widehat{W}^{(n)}(t) \triangleq \frac{1}{\sqrt{n}} W^{(n)}(nt)$

Theorem (Kingman (1961), Iglehart/Whitt (1970))

$$\widehat{W}^{(n)} \Rightarrow W^*,$$

where W^* is a Brownian motion with drift $-\gamma$, reflected at the origin so as to always be nonnegative.

Lead Times

- $L_1^{(n)}, L_2^{(n)}, \ldots$ – IID positive random variables. The lead times.
- $G(y)$ – Cumulative distribution function.

$$\mathbb{P}\left\{\frac{L_j^{(n)}}{\sqrt{n}} \leq y\right\} = G(y)$$

Customers are assigned lead times upon arrival, and lead times decrease at rate 1 per unit time thereafter. Delay grows like $\sqrt{n}$, so we must let lead times also grow like $\sqrt{n}$.

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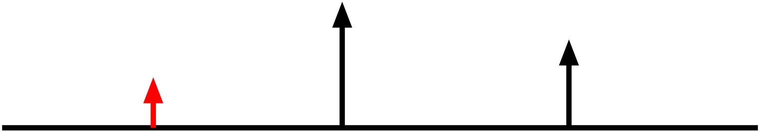
Earliest Deadline First (EDF) – Always serve customer with smallest lead time. Ties do not matter. Use pre-emption.

Problem: Determine the heavy traffic limit of the distribution of lead times of customers in queue.

Dynamics of lead times under EDF

$F^{(n)}(t)$ – Largest lead time of any customer who has ever been in service by time t, the frontier.



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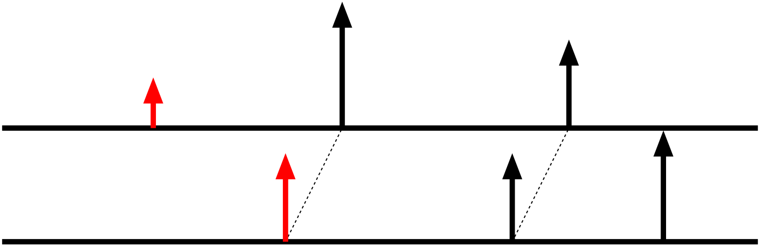

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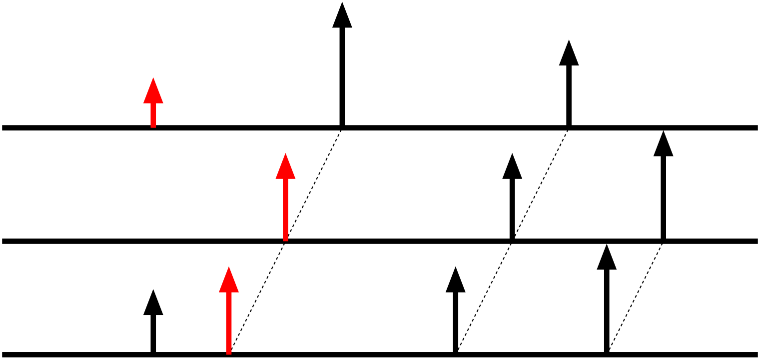

Workload and arrived-work measures

Let B be a Borel subset of $\mathbb{R}$. Define

$$\mathcal{W}^{(n)}(t)(B) \triangleq \left\{ \begin{array}{l} \text{Work associated with customers in} \\ \text{queue at time } t \text{ with lead times in } B. \end{array} \right\}$$

$$\mathcal{V}^{(n)}(t)(B) \triangleq \left\{ \begin{array}{l} \text{Work associated with customers arrived} \\ \text{by time } t \text{ with lead times in } B, \text{ whether} \\ \text{or not customer is still present at time } t. \end{array} \right\}$$

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Scaled processes

$$\widehat{\mathcal{W}}^{(n)}(t)(B) \triangleq \frac{1}{\sqrt{n}} \mathcal{W}^{(n)}(nt)(\sqrt{n}\,B),$$

$$\widehat{\mathcal{V}}^{(n)}(t)(B) \triangleq \frac{1}{\sqrt{n}} \mathcal{V}^{(n)}(nt)(\sqrt{n}\,B),$$

$$\widehat{F}^{(n)}(t) \triangleq \frac{1}{\sqrt{n}} F^{(n)}(nt).$$

Limiting lead-time distribution

Lemma (Crushing)

$$\widehat{\mathcal{W}}^{(n)}\big(-\infty, \widehat{F}^{(n)}\big] \Rightarrow 0.$$

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Corollary

For every $y \in \mathbb{R}$,

$$\widehat{\mathcal{W}}^{(n)}(t)(y,\infty) - \widehat{\mathcal{V}}^{(n)}(t)\big(y \vee \widehat{F}^{(n)}(t), \infty\big) \Rightarrow 0.$$

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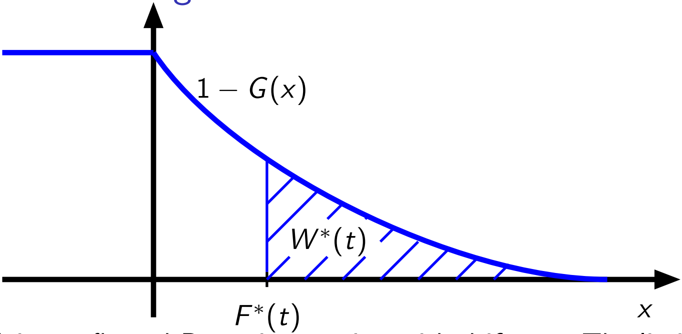
Theorem

For all $y \in \mathbb{R}$,

$$\widehat{\mathcal{V}}^{(n)}(t)(y,\infty) \Rightarrow H(y) \triangleq \int_y^\infty \big(1 - G(x)\big)\, dx.$$

Evolution of limiting workload measure



$W^*(t)$ is a reflected Brownian motion with drift $-\gamma$. The limiting scaled frontier is

$$F^*(t) = H^{-1}\big(W^*(t)\big).$$

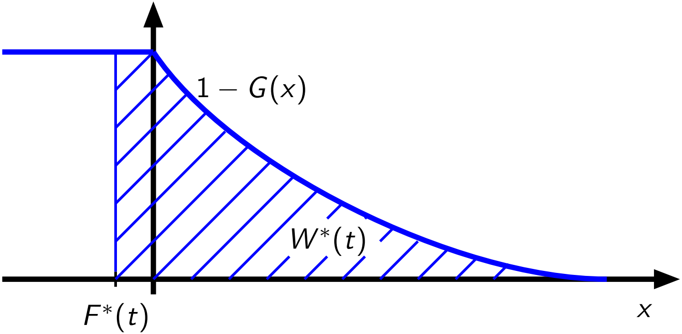
The limit of the measure-valued workload process $\widehat{\mathcal{W}}^{(n)}(t)$ has density $\big(1 - G(x)\big) I_{\{x \geq F^*(t)\}}$. We call this limiting measure-valued process

$$\mathcal{W}^*(t).$$

(Doytchinov, Lehoczky, Shreve (2000))

3. Real-Time Queues with Reneging

Customers are late in the limiting system when $F^*(t)$ is negative.



$$F^*(t) < 0 \Longleftrightarrow W^*(t) > H(0) = \int_0^\infty (1 - G(x))\, dx.$$

Theorem

If customers renege when their lead times reach zero, then the limiting scaled workload process is a

doubly reflected Brownian motion on $[0, H(0)]$ *with drift* $-\gamma$*.*

The limiting scaled workload measure is as before.

Ingredients of the proof

- $\mathcal{M}$ – The set of finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- $D_{\mathcal{M}}[0, \infty)$ – The set of cádlág functions taking values in $\mathcal{M}$. A sample path of the workload process, either scaled or unscaled, is an element of $D_{\mathcal{M}}[0, \infty)$.

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- $\Lambda\colon D_{\mathcal{M}}[0,\infty) \to D_{\mathcal{M}}[0,\infty)$

$$\Lambda(\mu)(t)(-\infty,y] \triangleq \left[\mu(t)(-\infty,y] - \sup_{0\le s\le t}\left(\mu(s)(-\infty,0] \wedge \inf_{s\le u\le t}\mu(u)(\mathbb{R})\right)\right]^+ .$$

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Set

$$\mathcal{U}^{(n)}(t) \triangleq \Lambda(\mathcal{W}^{(n)})(t).$$

Define

$$\begin{aligned}U^{(n)}(t) &\triangleq \mathcal{U}^{(n)}(\mathbb{R})(t)\\ &= W^{(n)}(t) - \sup_{0\le s\le t}\left[\mathcal{W}^{(n)}(s)(-\infty,0] \wedge \inf_{s\le u\le t} W^{(n)}(u)\right].\end{aligned}$$

The doubly-reflected Brownian motion U^*

Scale and pass to the limit:

$$U^*(t) \quad = \quad W^*(t) - \sup_{0 \le s \le t} \left[\mathcal{W}^*(s)(-\infty, 0] \wedge \inf_{s \le u \le t} W^*(u) \right].$$

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$$\mathcal{W}^*(s)(-\infty, 0] = \big(W^*(s) - H(0)\big)^+.$$

Therefore,

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Recall the double-reflection map for a scalar-valued process

$$\lambda(\phi)(t) \triangleq \phi(t) - \sup_{s \in [0,t]} \left[\big(\phi(s) - a \big)^+ \wedge \inf_{u \in [s,t]} \phi(u) \right].$$

The measures $\mathcal{W}_R^{(n)}$ and $\mathcal{U}^{(n)} = \Lambda(\mathcal{W}^{(n)})$

Let $D^{(n)}$ be the work that arrives to the reneging system ahead of the frontier and later reneges.

Lemma (Comparison)

For the unscaled processes, we have

$$0 \le U^{(n)}(t) - W_R^{(n)}(t) \le D^{(n)}(t).$$

For the scaled processes

$$\widehat{U}^{(n)}(t) \triangleq \frac{U^{(n)}(nt)}{\sqrt{n}}, \quad \widehat{W}_R^{(n)}(t) \triangleq \frac{W_R^{(n)}(nt)}{\sqrt{n}}, \quad \widehat{D}^{(n)}(t) \triangleq \frac{D^{(n)}(nt)}{\sqrt{n}},$$

we have the comparison

$$0 \le \widehat{U}^{(n)}(t) - \widehat{W}_R^{(n)}(t) \le \widehat{D}^{(n)}(t).$$

The measures $\mathcal{W}_R^{(n)}$ and $\mathcal{U}^{(n)} = \Lambda(\mathcal{W}^{(n)})$

Lemma (Crushing)
$\widehat{D}^{(n)} \Rightarrow 0.$

Theorem (Limit of reneging system)
$\widehat{\mathcal{W}}_R^{(n)} - \widehat{\mathcal{U}}^{(n)} \Rightarrow 0$, *or equivalently,* $\widehat{\mathcal{W}}_R^{(n)} \Rightarrow \Lambda(\widehat{\mathcal{W}}^*)$.