Positive Recurrence of Refecting Brownian Motion in 3 Dimensions

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(Abbreviated) Definition of semimartingale reflecting Brownian motion (SRBM) –

Process $Z(t)$, $t \geq 0$, on the nonregative orthant on $\mathbb{R}^d$, with

d-dim SRBM

d-dim. BM w/ drift vector θ, covariance matrix Γ (nondegenerate)

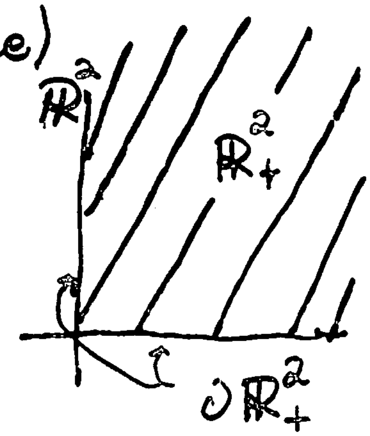
$$(1) \qquad \underset{\in \mathbb{R}^d_+}{Z(t)} = \underset{\in \mathbb{R}^d}{X(t)} + RY(t).$$

reflection matrix - determines direction of reflection on boundary

continuous, nondecreasing w/ $Y_i(0) = 0 \ \forall i$; $Y_i(t)$ only increases where $Z_i(t) = 0$

(X, Y adapted wrt filtered space)

- Z behaves like BM on $\mathring{\mathbb{R}}^d_+$ and reflects on $\partial \mathbb{R}^d_+$ according to R.



data of SRBM

- Existence & uniqueness of Z (in law) for given $\overbrace{(\theta, \Gamma, R)}$
iff $\underbrace{R \text{ is completely} - \mathcal{S}}_{\Rightarrow R_{ii} > 0 \ \forall i}$ (Reiman-Williams, Taylor-Williams).

Here, we will scale $\mathbb{R}^d$ so that $R_{ii} = 1$ for $i = 1, \dots, d$.

which θ, Γ, R

Question: When is Z positive recurrent?

Elementary requirement:

(2) R is nonsingular with $R^{-1}\theta < 0$
(El Kharroubi et al. (2000))

Some background:

- For all d, (2) is necessary & sufficient when R is an M-matrix (Harrison-Williams (1987)).

- For $d=2$, positive recurrent iff R is a P-matrix & (2) holds (K et al. (2000)).
each principal submatrix of R has positive determinant

- For $d \geq 3$, (2) is not sufficient (even when R is a P-matrix) (K et al. (2000)).
will discuss their example later

Here, will answer question when $d=3$.

For $d \geq 4$ – unknown.

Fluid paths: basics

Wish to reduce problem of positive recurrence to a deterministic problem.

Fluid path (y, z) satisfies deterministic analog of (1):

same θ, R as in (1)

(3) $z(t) = z(0) + \theta t + R\, y(t)$.

$\in \mathbb{R}^d_+$

(y, z) is associated w/ (θ, R)

continuous, nondecreasing w/ $y_i(0) = 0$ $\forall i$; $y_i(t)$ only increases where $z_i(t) = 0$

vocabulary:

A fluid path (FP) (y, z) is attracted to the origin if $z(t) \to 0$ as $t \to \infty$.

A FP (y, z) is divergent if $|z(t)| \to \infty$ as $t \to \infty$.

sufficient condition for positive recurrence

Theorem 1 (Dupuis-Williams (1994)). Let Z be a d-dimensional SRBM w/ data (θ, Γ, R). If every FP associated w/ (θ, R) is attracted to the origin, then Z is positive recurrent.

Linear fluid paths

A FP (y, z) of (3) is *linear* if $y(t) = ut$, $z(t) = vt$ for some $u, v \geq 0$.

LFP

For a LFP, (3) and its side conditions reduce to

$$(4) \qquad v = \theta + Ru, \quad u \cdot v = 0, \quad u, v \geq 0.$$

(LCP)

An LFP is *stable* if $v = 0$.
An LFP is *divergent* if $v \neq 0$.

Lemma 1. Suppose (2) holds and set $u^* = -R^{-1}\theta$. Then $(u^*, 0)$ is a stable LFP and all other LFPs are divergent.

Example of SRBM in $d=3$ that is not positive recurrent:

(Bernard - El Kharroubi (1991), K. et al. (2000))

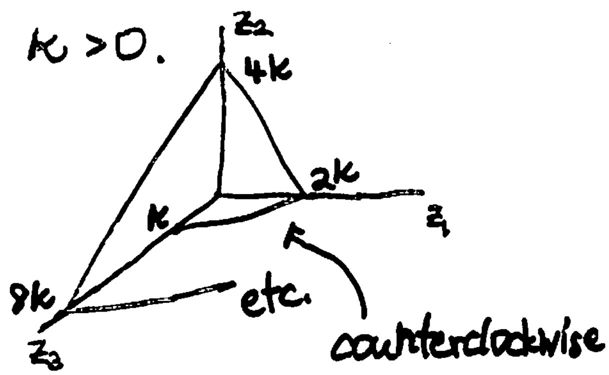
(5) $\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}$

with

$$z(0) = (0, 0, \kappa), \quad \kappa > 0.$$

Evolution of FP -

diagram 1



- FP is divergent.
- Using various estimates, K. et al. (2000) showed corresponding SRBM transient.
  for any nonsingular Γ

Can generalize (5) in a natural way so $z(t)$ moves piecewise linearly among the z_1, z_2, z_3 axes and diverges as above; denote this set of (θ, R) by C_1,

C_2 = corresponding (θ, R) in clockwise direction,
$C = C_1 \cup C_2$.

Set

$\beta(\theta, R)$ = scaling factor upon 1st return to z_3 axis.

For $(\theta, R) \in C$, SRBM is transient if $\beta(\theta, R) > 1$ and R is a P-matrix (K. et al (2000)).

Summary of results in 3 dimensions

stability result for FP

Theorem 2 (El Kharroubi et al. (2002)).

Suppose (2) and either

(a) $(\theta, R) \in C$ and $\beta(\theta, R) < 1$

or

LCP

(b) $(\theta, R) \notin C$ and (4) has a unique solution.

Then all FPs associated with (θ, R) are attracted to the origin.

By DW (1994), the corresponding SRBM is positive recurrent.

Note: proof is not entirely rigorous; alternative proof given in Dai-Harrison (2009).

Want to know behavior in converse direction. Given by following results:

For $(\theta, R) \in C$:

<u>Theorem 3</u> (B-Dai-Harrison (2009)). If $(\theta, R) \in C$ and $\beta(\theta, R) \geq 1$, then the SRBM is not positive recurrent.

For $(\theta, R) \notin C$, it suffices to consider the case where there is a divergent LFP.

Because of Theorem 2(b) and Lemma 1.

<u>Theorem 4</u>. (BDH (2009)). If there exists a divergent LFP, then the SRBM is not positive recurrent.

So, the conditions (a) and (b) in Theorem 2 give necessary and sufficient conditions for positive recurrence.

The proof of Theorem 3 is conceptual and not difficult.

The proof of Theorem 4 breaks into 5 cases, depending on the number of nonzero components of u and v for the LFP (u, v). A fair amount of work; some cases easier than others. Need to examine the asymptotic behavior of related BMs including hitting times of different boundaries.

Summary of proof of Theorem 3

For concreteness, use the Bernard - El Kharroubi example, with

$$(5') \quad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = -e, \quad R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}, \quad \Gamma = I.$$

label coordinates as R_{ij}

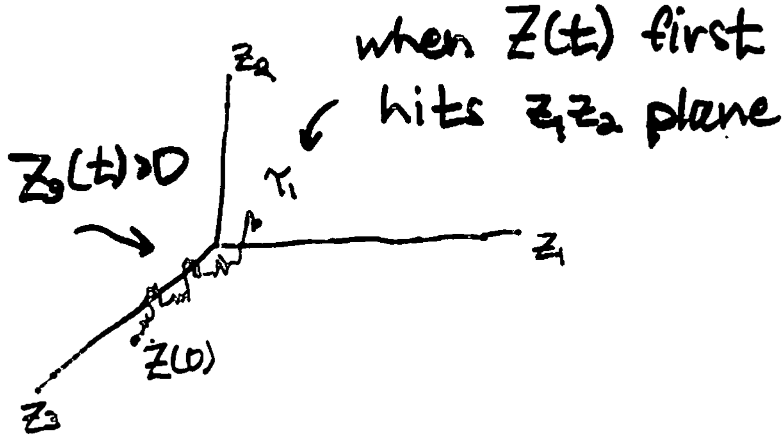
Also, assume $Z_1(0) \geq 0$, $Z_2(0) = 0$, $Z_3(0) > 0$.

(Argument holds for $(\theta, R) \in C$.)

In the spirit of diagram 1, set

$$(6) \quad \begin{aligned} \tau_1 &= \inf\{t > 0 : Z_3(t) = 0\} \\ \tau_2 &= \inf\{t > \tau_1 : Z_1(t) = 0\} \\ \tau_3 &= \inf\{t > \tau_2 : Z_2(t) = 0\} \\ &\vdots \end{aligned}$$

diagram 2



[no $Y_1(t)$ term in (7a) since $R_{21}=0$
no $Y_3(t)$ term in (7a) since $Z_3(t)>0$ on $t<\tau_1$]

Can check:

standard BMs

$$(7a)\quad Z_2(t) = B_2(t) - t + Y_2(t), \quad 0 \le t \le \tau_1,$$

$$(7b)\quad Z_3(t) = [B_3(t) - B_3(\tau_1)] - (t-\tau_1) + [Y_3(t) - Y_3(\tau_1)], \quad \tau_1 \le t \le \tau_2,$$

$$(7c)\quad Z_1(t) = [B_1(t) - B_1(\tau_2)] - (t-\tau_2) + [Y_1(t) - Y_1(\tau_2)], \quad \tau_2 \le t \le \tau_3,$$

$\vdots$

Rewriting (7a):

$$(8)\quad Y_2(t) = t + Z_2(t) - B_2(t) \quad \text{on } 0 \le t \le \tau_1.$$

On the other hand, note that for $u = \begin{pmatrix} 1/8 \\ 1/2 \\ 1/4 \end{pmatrix}$, one has

can always choose $u>0$ so ~~this~~ holds, for $\beta(\Theta, R) \ge 1$

$$(9)\quad u'R \ge e', \quad u'e = 1.$$

We wish to define a "distance" from $Z(t)$ to 0. Set

$$\xi(t) = u'Z(t).$$

will construct a martingale from this & compare w/ 1-d BM (which is not positive recurrent)

For $t \ge 0$,

$$\begin{aligned}
\xi(t) - \xi(0) &= u'\overset{(B_1(t),\, B_2(t),\, B_3(t))}{\overset{\|}{B(t)}} - \overset{=1}{u'e}\,t + \overset{\ge e'}{u'R}\,Y(t) \\
&\ge u'B(t) - t + e'Y(t).
\end{aligned} \tag{10}$$

On $t \le \tau_1$, this is, by (8),

no $Y_3(t)$ term

$$\begin{aligned}
&= u'B(t) - t + Y_1(t) + [t + Z_2(t) - B_2(t)] \\
&= \underbrace{u'B(t) - B_2(t)}_{\overset{\text{def}}{=}\ M(t)} + \underbrace{Y_1(t) + Z_2(t)}_{\overset{\text{def}}{=}\ A(t)}.
\end{aligned}$$

Define $M(t)$, $A(t)$ analogously on $(\tau_i, \tau_{i+1}]$. Then

$$\xi(t) - \xi(0) \ge M(t) + A(t) \quad \text{for all } t \ge 0.$$

$M(t)$ is a continuous martingale with quadratic variation

some $\gamma < \infty$

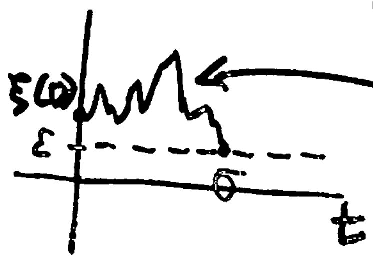
$$\langle M, M\rangle(t) - \langle M, M\rangle(s) \le \gamma(t-s) \quad \text{for } s \le t,$$

$A(t)$ is continuous with $A(t) \ge 0$ for all t.

For given $\varepsilon \in (0, \xi(0))$, set

$$\sigma = \inf\{t > 0 : \xi(0) + M(t) = \varepsilon\},$$
$$T = \inf\{t > 0 : \xi(t) = \varepsilon\}.$$

$\sigma \le T$



$M(t) + \xi(0)$
(which is $\le \xi(t)$)

A standard argument shows that

$$E[\sigma] = \infty.$$

same reasoning as for BM

Hence

$$E[T] = \infty.$$

This implies $Z(t)$ is not positive recurrent.