ORIE 633 Network Flows November 8, 2007

Problem Set 5

Due Date: November 20, 2007

1. In class, when considering the minimum-cost circulation problem we showed that we could cancel all negative-cost cycles in $O(m^2 n^2 \log(nC))$ time. When considering the generalized circulation problem, we claimed that we could cancel all flow generating cycles by using this algorithm. We do this by considering costs $c_{ij} = -\log \gamma_{ij}$ and cancelling negative-cost cycles. However, the costs c_{ij} are no longer integral, which was an assumption needed to prove the running time above. Assume that the gains γ_{ij} are ratios of integers that are bounded in absolute value by B. Show that in this case, the running time of the cycle cancelling algorithm we used in class is $O(m^2 n^3 \log(nB))$.

2. In the minimum-cost generalized circulation problem, we are given costs c_{ij} in addition to gains γ_{ij} and capacities u_{ij}. The goal is to find a *generalized circulation* g that minimizes $\sum_{(i,j)\in A} c_{ij} g_{ij}$. A generalized circulation is a generalized pseudoflow that has $e_i^g = 0$ for all $i \in V$.

   For the generalized flow problem we considered in class, we showed that the flow is maximum if and only if there are no generalized augmenting paths. In the case of the minimum-cost generalized circulation problems, the objects of interest are *unit gain cycles* and *bicycles*. A unit gain cycle C has $\gamma(C) = 1$. A bicycle has a flow-generating cycle C_1 connected by a path (possibly trivial) to a flow-absorbing cycle C_2.

   (a) Prove that a minimum-cost generalized circulation g is optimal if and only if there are no negative-cost unit gain cycles and no negative-cost bicycles in the residual graph G_g.

   (b) (Bonus) Suppose that as part of the input we are given b_i, and we must find a minimum-cost generalized pseudoflow such that $e_i^g = b_i$ for all $i \in V$. Determine optimality conditions as above for this problem (that is, prove that the flow is optimal if and only if there are no objects X in G_g.)

3. In this problem, we will consider a push/relabel style of algorithm for the generalized circulation problem we discussed in class. We say that an arc (i,j) is *admissible* if it has relabeled gain $\gamma_{ij}^{\mu} > 1$ and positive residual capacity. We say that a node i is active if it can reach the sink and it has positive excess (that is, $e_i^{h,\mu} > 0$). Now we allow a network G to specify some initial excess $e_i \geq 0$, so that for a flow h its excess at i is its initial excess plus the excess resulting from h: $e_i^{h,\mu} = e_i^{\mu} - \sum_{k:(i,k)\in A} h_{ik}^{\mu}$.

Push/relabel generalized circulation

Input: A network G with no flow-generating cycles, excesses $e_i \geq 0$, error parameter ϵ
Output: ϵ-optimal flow g

Set $b = (1+\epsilon)^{1/n}$ and round gains γ to powers of b as in Rounded Truemper
Let H be resulting network
$h \leftarrow 0$
Compute canonical labels μ
While there is an active node i
  If there is an admissible arc (i,j)
    Push: Send $\min(e_i^{h,\mu}, u_{ij}^{h,\mu})$ units of flow on (i,j), update h^μ
  Else
    Relabel: $\mu_i \leftarrow \mu_i / b^{1/n}$
Return g^μ, the interpretation of flow h^μ in G

(a) Prove that the algorithm maintains a flow h and labels μ such that $\gamma_{ij}^{\mu} \leq b^{1/n}$ for all residual arcs $(i,j) \in A_h$.

(b) Prove that during the course of the algorithm, the graph of admissible arcs is acyclic.

(c) Prove that during the course of the algorithm H_h has no flow-generating cycles.

(d) Prove that on termination, the algorithm outputs an ϵ-optimal flow.

(e) Prove that there are at most $O(\frac{1}{\epsilon} n^3 \log B)$ relabels per node.

(f) Prove that there are at most $O(\frac{1}{\epsilon} mn^3 \log B)$ saturating pushes.

(g) Prove that there are at most $O(\frac{1}{\epsilon} mn^4 \log B)$ nonsaturating pushes.

Assuming that we can implement both the push and relabel operations in $O(\log n)$ time, we can thus find an ϵ-optimal flow in $O(\frac{1}{\epsilon} mn^4 \log B \log n)$ time.

(h) Using the above algorithm as a subroutine, and assuming we can cancel flow-generating cycles in $O(mn^3 \log(nB))$ time, obtain an $O(m^2 n^4 \log^2 B \log n)$ time algorithm for the maximum generalized flow problem.

4. (Bonus) Suppose you are given an instance of the generalized flow problem that in addition has costs on the arcs. Consider a Bellman-Ford style algorithm that tries to determine the minimum-cost method for sending a unit of flow from each vertex i to the sink t. Let y_i^k be the cost of sending a unit of flow along a walk of at most k arcs. Thus we set $y_t^0 = 0$, $y_i^0 = \infty$ for $i \neq t$, and update

$$y_i^k = \min(y_i^{k-1}, \min_{(i,j)\in A} (c_{ij} + \gamma_{ij} y_j^{k-1})).$$

Assume that there are no negative-cost bicycles and no negative-cost unit gain cycles in the graph. Prove that in $O(mn)$ time you can detect whether there is a negative-cost GAP.