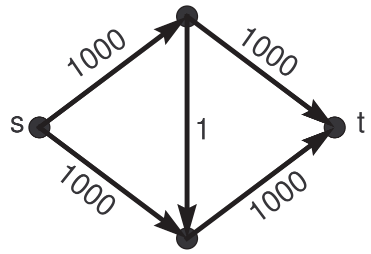
ORIE 633 Network Flows	August 30, 2007

Problem Set 1

Due Date: September 13, 2007

1. In this problem, we will consider other possibilities augmenting paths that can give rise to polynomial time algorithms for the maximum s-t flow problem.

   (a) Consider the max s-t flow problem shown below, where the numbers on the arcs represent capacities. Show that an algorithm that chooses arbitrary augmenting paths from the residual graph does not run in polynomial time.

   (b) Consider an algorithm that always chooses the *shortest* augmenting path in the residual graph (in terms of number of arcs). Prove that the number of augmentations is bounded by $O(nm)$. (One possible proof would show that the length of the shortest path never decreases after an augmentation, and that for all augmentations with path length k, the set of edges in augmenting paths of length k decreases with each augmentation).

2. In class, we showed that the maximum flow problem can be used to solve the problem of determining when a baseball team k cannot achieve the most wins in its division (i.e. the team is *eliminated*). Let w_i be the number of wins a baseball team has, let g_{ij} be the number of games that teams i and j have remaining to play, and let $g_i = \sum_j g_{ij}$.

   (a) Prove that if team k is eliminated, and $w_k + g_k \geq w_i + g_i$, then team i is also eliminated. (Hint: what condition did we use to show that team k was eliminated?)

   (b) Prove that if n is the total number of teams in the division, then with $O(\log n)$ executions of a maximum flow algorithm, we can determine which teams in the division have been eliminated and which have not.

3. Let $G = (V, A)$ be a directed graph, with bounds $0 \leq l_{ij} \leq u_{ij}$ for all $(i, j) \in A$. We say that f is a feasible *circulation* if for all $i \in V$,

$$\sum_{j:(j,i)\in A} f_{ji} - \sum_{j:(i,j)\in A} f_{ij} = 0,$$

and for all $(i,j) \in A$, $l_{ij} \leq f_{ij} \leq u_{ij}$.

For $S \subset V$, define $\delta^+(S) = \{(i,j) \in A : i \in S, j \notin S\}$ and $\delta^-(S) = \{(i,j) \in A : i \notin S, j \in S\}$. Also, for a subset $A' \subseteq A$ of arcs, define $u(A') = \sum_{(i,j)\in A'} u_{ij}$ and $l(A') = \sum_{(i,j)\in A'} l_{ij}$.

Prove that there exists a feasible circulation f if and only if for all subsets $S \subset V$, $S \neq \emptyset$, $u(\delta^+(S)) \geq l(\delta^-(S))$.

4. Consider the following problem.

- **Input:**
  - A directed graph $G = (V, A)$
  - A source $s \in V$ and a sink $t \in V$, $s \neq t$
  - Integer capacities $u_{ij} \geq 0$ for all $(i,j) \in A$
  - A positive integer k
- **Goal:** Find the maximum amount of flow that can be sent from s to t if flow is sent on exactly k paths and each path must send the same amount of flow.

We claim that a max flow/min cut theorem can be shown for this problem for a suitable definition of cut. For a cut S, $s \in S$, consider a bin packing problem in which there is a bin of capacity u_{ij} for each arc $(i,j) \in \delta^+(S)$. Let λ be the maximum item size such that k items of size λ can be packed in these bins. Then we define the capacity of cut S, $\hat{c}(S)$, to be $k\lambda$.

Prove that the maximum s-t flow of the type above is equal to the minimum capacity s-t cut as defined above; i.e. $\min_{S\subseteq V: s\in S, t\notin S} \hat{c}(S)$.