PRICING OPTIONS FROM THE POINT OF VIEW OF A TRADER

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Abstract. This paper is a contribution to the pricing and hedging of options in a market where the volatility is stochastic. The new concept of relative indifference pricing is further developed. This relative price is the price at which an option trader is indifferent to trade in an additional option, given that he is currently holding and dynamically hedging a portfolio of options. We find that the appropriate volatility risk premium depends on the trader's risk aversion coefficient and his portfolio position before selling or buying the additional option. We suggest two asymptotic expansions which relate the volatility risk premium to the Vega of the option portfolio. This approach provides a tool for traders to (i) integrate option pricing with risk management and (ii) quote competitive prices that depend on their aggregate risk exposure.

Keywords: stochastic volatility, incomplete markets, relative indifference pricing, risk management, indifference hedge, Vega, Volga, Vanna.

1. Introduction

Thirty years after its discovery, the Black and Scholes formula is still the only universally accepted formula for option pricing. Although option traders are fully aware that its basic assumptions are flawed, it remains the benchmark model and all other models are usually viewed as its adjustments or corrections. One of the most popular such deviation is to assume that the volatility is stochastic. Such models were first introduced by Hull and White (1987), Stein and Stein (1991) and Heston (1993). Although this approach successfully explains features such as the implied volatility smile observed in the markets and the heteroskedastic nature of stock returns, it has one important drawback. Because of the incompleteness induced by the fact that stochastic volatility cannot be traded, the classical dynamic replication argument of Black and Scholes breaks down. Indeed, there are infinitely many arbitrage-free pricing measures to pick from. Selecting the right pricing measure is equivalent to specifying the so-called "market price of volatility risk" or "volatility risk premium".

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In most practical applications, the issue of specifying a volatility risk premium is essentially circumvented by ignoring the historical dynamics of the volatility process and working directly with its dynamics under the pricing measure. For instance, Hagan et al. (2002) postulate a particular volatility process under the pricing measure and calibrate its parameters to market prices. In Avellaneda and Paras (1996), the authors assume that the volatility process lies within an upper and lower bound under all admissible pricing measures. This band translates directly into prices and hedging strategies for vanilla options, portfolios of options and exotic options.

However, if one wants to use the statistical or historical distribution of the volatility process, one needs to introduce an additional criterion to select the "optimal" pricing measure and the related hedging strategy. Some of the most studied criteria have been to minimize the variance of the tracking error (Föllmer and Sondermann (1986)), maximize the probability of a successful hedge (Föllmer and Leukert (1999)) or minimize the shortfall risk (Föllmer and Leukert (2000)).

Alternatively, utility functions can be introduced through the notion of indifference pricing first proposed by Hodges and Neuberger (1989). The indifference price is the price at which an expected utility maximizing agent would be indifferent between (i) selling and partially hedging an option and (ii) abstaining from such a trade. This approach is closer in spirit to ours, and has been applied to stochastic volatility models in Sircar and Zariphopoulou (2005). The indifference price follows a nonlinear valuation rule which, in the limit as the amount of options purchased/sold goes to zero, converges to a linear price. This so-called "fair" price limit was suggested by Davis (1999) in the context of basis risk as a source of incompleteness. In the case of agents with exponential utility, this linear pricing rule is equivalent to arbitrage-free valuation under the minimal entropy martingale measure. This important connection was analyzed in Frittelli (2000), Rouge and El Karoui (2000) and Delbaen et al. (2002) under very general dynamics for the stock price. Ilhan, Jonsson and Sircar (2004) discuss the relation between indifference pricing and portfolio optimization and provide various asymptotic expansions.

In this paper, we modify the original definition of indifference prices and develop the notion of *relative indifference prices* for options when the volatility is stochastic. This concept was first proposed by Musiela and Zariphopoulou (2001b, 2004) for the valuation of claims written on non-traded assets in a continuous and discrete setting, respectively. Herein, we consider a general diffusion model for stochastic volatility and develop a framework for pricing options relative to an arbitrary book of options.

The relative indifference price is the price that makes a utility maximizing agent indifferent between (i) dynamically hedging a large book of options and (ii) dynamically hedging the book of options *plus* the option to be priced. Unlike the approaches mentioned above, which consider each option *in isolation*, this new approach views the pricing problem from the point of view of a trader or market maker who is maximizing his utility by optimally trading a book of options. Our main result is that the trader should use a volatility risk premium that depends on the sensitivity of his entire option book with respect to the volatility, also known as the portfolio "Vega".

When quoting a price for a marginal option, the trader must take into account his portfolio position at the moment of selling or buying this option. This observation is at the heart of the relative indifference price's two most striking features. First, it provides a link between option pricing and volatility risk management. Indeed, this mechanism gradually sets *limits on the volatility risk*, because as the trader accumulates more and more volatility risk in his portfolio, the price quotes for risk-reducing options rise and those for risk-enhancing options fall. Second, it is a tool that allows traders to quote *competitive prices*. Since all financial institutions have different option books and risk exposures, our mechanism will highlight the type of options where the trader has a comparative advantage.

Recent econometric studies, such as Bakshi and Kapadia (2003) and Sarwar (2002) have reported a negative volatility risk premium that depends on the option's moneyness and on the level of volatility. The relative indifference pricing method provides a specific functional form for the volatility risk premium that is consistent with this empirical fact. Indeed, if the market makers are net sellers of calls and puts, the Vega of their option book will be negative (i.e. short volatility) and our theory predicts that this will contribute towards a negative volatility risk premium (see (3.5)).

The paper is organized as follows. In Section 2, we introduce the stochastic volatility model and review the indifference valuation approach. As the size of the trade tends to zero, the indifference price is given by the expected payoff under the minimal entropy martingale measure. We also extend the indifference approach to portfolios of options.

In Section 3, we define the *relative indifference price*. Once again, taking the limit as the size of the trade tends to zero, this new price is given by a linear pricing rule under an equivalent martingale measure. We derive the associated volatility risk premium and find that it depends on the indifference price of the trader's existing option book. We also present explicit expressions for the optimal dynamic hedging strategy for the option book.

In Section 4, we suggest two asymptotic expansions for the relative indifference price. The small risk aversion expansion provides an adjustment to the minimal entropy price. The small correlation/ slow mean reversion expansion provides adjustments to the Black and Scholes price.

In Section 5, we present a numerical procedure for the small correlation/ slow mean reversion expansion. This allows us to obtain implied volatility surfaces under various scenarios of option books held by the trader. Our approximation formula, see (5.1), suggests the importance of keeping track of the option book's Vega over future time intervals. A variation of this methodology known as "Vega bucketing" is already used by the risk management departments that oversee trader activity.

2. Indifference pricing

The Market Model. We consider a dynamic market setting with two assets, a riskless bond B yielding constant interest rate $r = 0$ and a stock S. The stock price is modeled as a diffusion process satisfying, for times $s \geq t$,

$$
\begin{cases} dS_s = \mu S_s ds + \sigma_s S_s dW_s \\ S_t = S \geq 0 \end{cases} \tag{2.1}
$$

where the drift μ is constant and the volatility σ_t of the stock returns is a correlated diffusion satisfying

$$
\begin{cases} d\sigma_s = b(\sigma_s, s)ds + a(\sigma_s, s)(\rho dW_s + \bar{\rho} dW_s^{\perp}) \\ \sigma_t = \sigma \end{cases} \tag{2.2}
$$

with $-1 < \rho < 1$ and $\bar{\rho} = \sqrt{1-\rho^2}$. W_s and $W_s^{\perp}$ are two independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ where $\mathcal{F}_s$ is the augmented σ-algebra generated by the Brownian motions. The above market model is a standard way of incorporating heteroskedastic behavior (i.e. a stochastic volatility that is correlated to the stock's returns) in continuous time. In order to ensure that (2.1) and (2.2) have unique solutions and that the PDE introduced in the sequel have classical solutions, we will make the following assumptions throughout.

Assumptions
(i) The process σ_t is bounded above and below away from zero.
(ii) The drift coefficient $b(\sigma, s)$ is Lipschitz continuous.
(iii) The diffusion coefficient $a(\sigma, s)$ is smooth, bounded above and below away from zero.
(iv) All option payoffs $g(S)$ and $g_i(S)$ are smooth and bounded.

Note that assumptions (i), (ii) and (iii) can be relaxed to accommodate for several popular stochastic volatility models such as the Stein and Stein (1991) and Heston (1993) models (see Benth and Karlsen (2003)). Weakening assumption (iv) to include call options (which are unbounded and non-smooth) is addressed in Ilhan and Sircar (2004) and Fouque et. al. (2003).

It follows from classical arbitrage-free arguments that the price of any European option with payoff $g(S_T)$ can be written as the expected payoff $\nu_t(g) = E_{\mathbb{Q}}[g(S_T)|S_t = S, \sigma_t = \sigma]$ under some equivalent martingale measure $\mathbb{Q}$. Under this, not necessarily unique, measure $\mathbb{Q}$, the state variables follow the modified dynamics

$$
\begin{cases} dS_s = \sigma_s S_s dW_s \\ S_t = S \geq 0 \end{cases} \tag{2.3}
$$

and

$$
\begin{cases} d\sigma_s = b^*(\sigma_s, s)ds + a(\sigma_s, s)(\rho dW_s + \bar{\rho} dW_s^{\perp}) \\ \sigma_t = \sigma \end{cases} \tag{2.4}
$$

where

$$
b^*(\sigma, t) = b(\sigma, t) - a(\sigma, t)\Lambda_t.
$$

The stochastic process Λ_t is called the *volatility risk premium* and affects the drift of the volatility process. It is often written as

$$
\Lambda_t = \rho \frac{\mu}{\sigma_t} + \bar{\rho}\lambda_t
$$

where the first term represents the effect of the market price of W_t risk and the second term conveys the effect of the market price of $W_t^{\perp}$ risk. In fact, it follows from the Girsanov theorem that there is a one-to-one correspondence between the stochastic processes λ_t such that $\int_t^T \lambda_s^2 ds < \infty$ and the generic martingale measure $\mathbb{Q}$. The option pricing problem thus reduces to the selection of the volatility risk premium.

An approach often used by practitioners is to simply ignore the historical measure $\mathbb{P}$, directly specify a particular dynamics for equation (2.4) and estimate the parameters implied by the market prices of some liquid options. For instance Heston (1994) suggests $b^*(\sigma,t) = \beta\left(\frac{M}{\sigma} - \sigma\right)$ and $a(\sigma,t) = \alpha$ while Hagan et al. (2002) propose $b^*(\sigma,t) = 0$ and $a(\sigma,t) = \alpha\sigma$. These particular functional forms may seem like a natural choice for the function $b(\sigma,t)$, since they both model a positive volatility process. However, the arbitrage-free theory makes no prediction on the possible form of λ_t and the forms suggested above may be inappropriate for the drift term $b^*(\sigma,t)$.

In order to specify the structure of the market price of volatility risk, we will need to make further assumptions. Namely, we need to select a λ_t that is consistent with the trader's optimal risk monitoring strategies. In the sequel, we relate pricing under various measures to optimal investment and hedging strategies for a utility maximizing investor. This will provide us with an economic justification for pricing under these martingale measures.

Introducing Utility Functions. First, we briefly review the indifference pricing problem in the setting of stochastic volatility, along the lines of Sircar and Zariphopoulou (2005).

The investor starts with initial wealth x at time $t \geq 0$, and dynamically adjusts the monetary amount invested in the stock π_s for $s > t$. Since $r = 0$, the total wealth solves the following stochastic differential equation

$$\begin{cases} dX_s = \mu\pi_s ds + \sigma_s\pi_s dW_s \\ X_t = x, \quad x \in \mathcal{R}. \end{cases}$$

The control variable π_s is called admissible if it is $\mathcal{F}_s$-measurable and satisfies the integrability condition $E_{\mathbb{P}} \int_t^T \sigma_s^2 \pi^2 ds < \infty$. The set of admissible policies is denoted by $\mathcal{A}$. Note that the case of deterministic, non-zero interest rate may be handled by straightforward scaling arguments and is not discussed.

Next, we introduce the two optimization problems that give rise to the indifference price of an option. Throughout, we assume that the individual preferences are modelled by an exponential utility function

$$U(x) = -e^{-\gamma x}, \quad x \in \mathcal{R}$$

with risk aversion parameter $\gamma > 0$. We also assume that this preference functional is not affected by the quantity of derivatives that is bought or sold.

The first problem is the classical Merton problem of optimal investment with stochastic volatility with value function

$$V(x,\sigma,t,T) = \max_{\pi\in\mathcal{A}} E\left(-e^{-\gamma X_T} | X_t = x, \sigma_t = \sigma\right). \tag{2.5}$$

The second problem is an optimal investment problem for an agent who has bought a European type option with payoff $g(S_T)$. Its value function is

$$V^g(x,\sigma,S,t,T) = \max_{\pi\in\mathcal{A}} E\left(-e^{-\gamma(X_T+g(S_T))}|X_t = x, \sigma_t = \sigma, S_t = S\right).$$

The indifference price represents the amount by which an agent needs to be compensated to be indifferent between the two investment problems described above.

Definition 1. The indifference price $\nu_t(g)$ of a European claim with payoff $g(S_T)$ is given by

$$V(x+\nu_t(g),\sigma,t,T) = V^g(x,S,\sigma,t,T).$$

After deriving the Hamilton-Jacobi-Bellman equations that the functions V and V^g solve, and substituting appropriately chosen ansatz solutions, one can derive a quasilinear partial differential equation for the indifference price. This solution technique is presented in detail in Sircar and Zariphopoulou (2005) and we only quote their result (Theorem 2.9) below.

To facilitate the presentation, we suppress the arguments of the diffusion coefficients and introduce the operators

$$\mathcal{L}^\sigma = \frac{1}{2}a^2\frac{\partial^2}{\partial\sigma^2} + (b-\rho a\frac{\mu}{\sigma})\frac{\partial}{\partial\sigma}$$

and

$$\mathcal{L}^{S,\sigma} = \frac{1}{2}a^2\frac{\partial^2}{\partial\sigma^2} + \rho a\sigma S\frac{\partial^2}{\partial S\partial\sigma} + \frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2} + (b-\rho a\frac{\mu}{\sigma})\frac{\partial}{\partial\sigma}.$$

Theorem 2.1. *The indifference price*

$$\nu_t(g) = h(S,\sigma,t)$$

is the unique $C^{2,2,1}(\mathbb{R}\times\mathbb{R}\times[t,T])$ *bounded solution of the pricing equation*

$$\left\{\begin{array}{l} h_t + \mathcal{L}^{S,\sigma}h + \bar\rho^2a^2\phi_\sigma h_\sigma - \frac{1}{2}\gamma\bar\rho^2a^2h_\sigma^2 = 0 \\ h(S,\sigma,T) = g(S), \end{array}\right. \tag{2.6}$$

where $\phi(\sigma,t,T)$ *solves*

$$\left\{\begin{array}{l} \phi_t + \mathcal{L}^\sigma\phi + \frac{1}{2}\bar\rho^2a^2\phi_\sigma^2 = \frac{\mu^2}{2\sigma^2} \\ \phi(\sigma,T,T) = 0. \end{array}\right. \tag{2.7}$$

The pricing mechanism described above has several notable properties. First, it does not depend on the investor's wealth x. Second, it is nonlinear, as can be readily seen from the quadratic gradient term, $\frac{1}{2}\gamma\bar\rho^2a^2h_\sigma^2$, in the pricing equation (2.6). It is natural to consider the price per unit as the trade size tends to zero

$$p_t(g) \triangleq \lim_{\alpha\to 0}\nu_t(\alpha g)/\alpha$$

which converges monotonically to the so-called limiting 'fair' price of Davis (1999) and solves a linear pricing rule given by (2.8). Indeed, by multiplying the payoff by α, dividing equation (2.6) by α and taking the limit as $\alpha\to 0$, we find that the nonlinear term drops out.

Corollary 2.2. *The limiting fair price of an option is given by*

$$p_t(g) = h^*(S, \sigma, t)$$

where h^* solves

$$\begin{cases} h^*_t + \mathcal{L}^{S,\sigma} h^* + \bar{\rho}^2 a^2 \phi_\sigma h^*_\sigma = 0 \\ h^*(S, \sigma, T) = g(S). \end{cases} \tag{2.8}$$

The price is given probabilistically by $p_t(g) = E_{\tilde{\mathbb{Q}}}[g(S_T)|S_t = S, \sigma_t = \sigma]$ where $\tilde{\mathbb{Q}}$ is the martingale measure whose volatility risk premium is

$$\tilde{\Lambda}(\sigma, t) = \rho \frac{\mu}{\sigma} - \bar{\rho}^2 a \phi_\sigma$$

Remark 2.1. The measure $\tilde{\mathbb{Q}}$ is known as the minimal entropy measure and $p_t(g)$ as the minimal entropy price. The important connection between exponential utility and entropy is further discussed in Rouge and El Karoui (2000) and Delbaen et al. (2002). In the above setting, the minimal entropy price can also be interpreted as the $\gamma \to 0$ limit of the indifference price. This can be readily checked by formally setting γ to zero in equation (2.6).

Portfolios of Options. We conclude this section by generalizing the indifference pricing method to portfolios of options. First consider an agent who has accumulated a portfolio

$$P = \sum_{i=1}^{n} g_i(S_{t_i})$$

of European options maturing at various future times $t \le t_1 \le t_2 \le \ldots \le t_n$ and with payoffs $g_i(\cdot)$. The value function of a trader who is holding (and is dynamically hedging) this portfolio of options is given by

$$V^P(x, \sigma, S, t, t_n) = \max_{\pi \in \mathcal{A}} E\left(-e^{-\gamma(X_{t_n} + \sum_{i=1}^{n} g_i(S_{t_i}))} | X_t = x, \sigma_t = \sigma, S_t = S\right). \tag{2.9}$$

The indifference price of the portfolio is given by

$$V(x + \nu_t(P), \sigma, t, t_n) = V^P(x, S, \sigma, t, t_n) \tag{2.10}$$

and we obtain the following result, which is a natural generalization of Theorem 2.1.

Theorem 2.3. *The indifference price of the portfolio P,*

$$\nu_t(P) = H(S, \sigma, t)$$

is the unique $C^{2,2,1}(\mathbb{R} \times \mathbb{R} \times [t, T])$ bounded solution of the pricing equation

$$\begin{cases} H_t + \mathcal{L}^{S,\sigma} H + \bar{\rho}^2 a^2 \phi_\sigma H_\sigma - \frac{1}{2} \gamma \bar{\rho}^2 a^2 H_\sigma^2 = \sum_{i=1}^{n-1} g_i(S) \cdot \delta(t - t_i) \\ H(S, \sigma, t_n) = g_n(S). \end{cases} \tag{2.11}$$

where $\phi(\sigma, t, t_n)$ solves (2.7).

In equation (2.11) above, we introduce the Dirac function δ for notational convenience. Formally, it describes the pasting condition required at each time t_i, i.e. $H(S,\sigma,t_i) = H(S,\sigma,t_i^+) + g_i(S)$. The proof of the above result can be found in the Appendix.

3. Relative indifference pricing

The Pricing Mechanism. We now define the notion of *relative indifference pricing*, which will model the behavior of an agent who has *already* undertaken a significant amount of volatility risk. This concept is meant to model the fact that at the moment of sale, traders are holding (and are dynamically risk monitoring) a portfolio of options with payoff P. Much like in the standard indifference pricing method, we will solve two stochastic optimization problems and define the relative price in terms of the two value functions. To remain consistent with the previous section's results, we introduce the new terminal time horizon $t_{\max} = \max(t_n, T)$, in terms of which the value functions will be defined.

The first problem is one of optimal investment for an agent who is holding a portfolio of options with payoff $P = \sum_{i=1}^n g_i(S_{t_i})$. Its value function is

$$V^P(x,\sigma,S,t;t_{\max}) = \max_{\pi\in\mathcal{A}} E\left(-e^{-\gamma(X_{t_{\max}}+\sum_{i=1}^n g_i(S_{t_i}))} | X_t = x, \sigma_t = \sigma, S_t = S\right).$$

The second problem is one of optimal investment for an agent who is holding a portfolio of options with payoff $P = \sum_{i=1}^n g_i(S_{t_i})$ and has bought an additional option with payoff $g(S_T)$, maturing at time T. Its value function is

(3.1)

$$V^{P+g}(x,\sigma,S,t;t_{\max}) = \max_{\pi\in\mathcal{A}} E\left(-e^{-\gamma(X_{t_{\max}}+\sum_{i=1}^n g_i(S_{t_i})+g(S_T))} | X_t = x, \sigma_t = \sigma, S_t = S\right).$$

The relative indifference price represents the amount by which an agent needs to be compensated to be indifferent between the two investment problems.

Definition 2. The relative indifference price $\nu_t(g|P)$ of the claim $g(S_T)$, given that the agent is holding a portfolio of options P, is defined as the amount $\nu_t(g|P)$ for which

$$V^P(x+\nu_t(g|P), S, \sigma, t; t_{\max}) = V^{g+P}(x, S, \sigma, t; t_{\max}) \tag{3.2}$$

at all wealth levels $x \in \mathbb{R}$. Note that this reduces to the standard indifference price when $P = 0$.

An important feature of relative indifference prices is their *additivity* in terms of the payoffs. Indeed, if we apply the portfolio indifference relation (2.10) to both sides of (3.2), we obtain the following result.

Proposition 3.1. *The relative indifference price $\nu_t(g|P)$ of an option with payoff $g(S_T)$ given that a portfolio $P = \sum_{i=1}^n g_i(S_{t_i})$ has been written satisfies*

$$\nu_t(P+g) = \nu_t(P) + \nu_t(g|P)$$

where $\nu_t(P)$ and $\nu_t(P+g)$ are the traditional indifference prices of P and $P+g$, respectively.

By using this property, we may express the relative indifference price as the solution of a quasilinear PDE. The proof of the following result can be found in the Appendix.

Theorem 3.2. *The relative indifference price is given by*

$$\nu_t(g|P) = f(S,\sigma,t)$$

where f is the unique $C^{2,2,1}(\mathbb{R}\times\mathbb{R}\times[t,T])$ bounded solution of the relative pricing equation

$$\left\{\begin{array}{l} f_t + \mathcal{L}^{S,\sigma} f + \bar{\rho}^2 a^2 \phi_\sigma f_\sigma - \gamma\bar{\rho}^2 a^2 H_\sigma f_\sigma - \frac{1}{2}\gamma\bar{\rho}^2 a^2 f_\sigma^2 = 0 \\ f(S,\sigma,T) = g(S). \end{array}\right. \tag{3.3}$$

Herein $\phi(\sigma,t;t_{\max})$ solves equation (2.7) *and the indifference price of the portfolio $H(S,\sigma,t)$ solves* (2.11).

Much like in our analysis of traditional indifference prices, we will be interested in the fair price limit of Davis. The limit as the amount of options α becomes small is considered, because our aim is to quote prices for a *marginal option*, given that we have already sold a large porfolio of options. Surprisingly, in the case of relative indifference pricing, the fair price does *not* coincide with the $\gamma\to 0$ or risk neutral limit. We now state this section's main result, which identifies the market price of volatility risk for the relative limiting fair price.

Theorem 3.3. *Let $p_t(g|P)$ be the relative limiting fair price defined by*

$$p_t(g|P) = \lim_{\alpha\to 0}\frac{\nu_t(\alpha g|P)}{\alpha}.$$

Then $p_t(g|P) = f^*(S,\sigma,t)$ where f^* solves

$$\left\{\begin{array}{l} f_t^* + \mathcal{L}^{S,\sigma} f^* + \bar{\rho}^2 a^2 \phi_\sigma f_\sigma^* - \gamma\bar{\rho}^2 a^2 H_\sigma f_\sigma^* = 0 \\ f^*(S,\sigma,T) = g(S). \end{array}\right. \tag{3.4}$$

The price is given probabilistically by $p_t(g|P) = E_{\hat{\mathbb{Q}}}[g(S_T)]$ where $\hat{\mathbb{Q}}$ is the martingale measure whose volatility risk premium is

$$\hat{\Lambda}(S,\sigma,t) = \rho\frac{\mu}{\sigma} - \bar{\rho}^2 a\phi_\sigma + \gamma\bar{\rho}^2 a H_\sigma(S,\sigma,t). \tag{3.5}$$

where H_σ is the indifference Vega of the existing option book.

The Risk Monitoring Strategy. Since we are in an incomplete market setting, there is no perfect dynamic replication strategy for options. We will therefore consider a partial hedge or a *risk monitoring strategy* for the option book that is optimal in the sense of maximal expected utilities. Recall that the indifference price concept is defined in terms of the optimal investment in the spot process with and without the option, so it is natural to consider the risk monitoring strategy as the difference of the two underlying investment policies. In other words, the indifference hedge of the option book is expected to be the difference

$$\Pi_t(P) \triangleq \pi_t^P - \pi_t^0,$$

where π_t^0 and π_t^P are the optimal controls in problems (2.5) and (2.9), respectively. We refer the reader to the Appendix for the solutions to these two problems, and for the proof of the following result.

Theorem 3.4. *The optimal risk monitoring strategy for the entire option book is given by,*

$$\Pi_t(P) = -S_t H_S(S_t, \sigma_t, t) - \rho \frac{a(\sigma_t, t)}{\sigma_t} H_\sigma(S_t, \sigma_t, t) \tag{3.6}$$

where H solves the nonlinear equation (2.11).

Remark 3.1. The optimal hedge consists of two components: the *Delta* hedge, SH_S, of the indifference price and a *Vega* hedge, $\rho \frac{a}{\sigma} H_\sigma$, which is a partial hedge to the changes in volatility, using the spot price as a correlated hedging instrument. Naturally, if the stock and the volatility are uncorrelated, the stock will be of no use in hedging volatility risk, and the trader will use the naive Delta hedging strategy.

Unlike in the complete market setting, the payoff of the option cannot be expressed simply in terms of the option price and the proceeds obtained from the risk monitoring strategy presented above. As can be seen in (3.12) below, there is a residual risk associated with our strategy which will cause us to over or under shoot the option payoffs.

The optimal wealth process of the classical Merton problem (2.5) solves

$$dX_s^{0,*} = \mu \pi_s^0 ds + \sigma_s \pi_s^0 dW_s \tag{3.7}$$

with $X_t^{0,*} = x$ and the optimal wealth process of the problem (2.9) compensated by the indifference price of the portfolio book is

$$dX_s^{P,*} = \mu \pi_s^P ds + \sigma_s \pi_s^P dW_s \tag{3.8}$$

with $X_t^{P,*} = x - H(S, \sigma, t)$. Following the methodology developed by Musiela and Zariphopoulou (2001a) (see, also Musiela and Zariphopoulou (2004) and Stoikov and Zariphopoulou (2004)), we first define the process

$$L_s = X_s^{P,*} - X_s^{0,*}$$

for $t \leq s \leq T$ to be the *residual optimal wealth.* Subtracting (3.7) from (3.8), we obtain the dynamics

$$dL_s = \Pi_t(P) \frac{dS_s}{S_s} \tag{3.9}$$

with $L_t = -H(S, \sigma, t)$. This quantity represents the proceeds obtained from the indifference risk monitoring strategy in (3.6). We next consider the *residual risk process*

$$R_s = H(S_s, \sigma_s, s) + L_s \tag{3.10}$$

with $R_t = 0$ to be the amount by which this strategy misses the option payoffs at times $s \in [t, T]$. Applying Ito's formula to (3.10) and using the equation (2.11) that

$H(S,\sigma,t)$ solves, we obtain that

$$dR_s = \left(\frac{1}{2}\gamma\bar{\rho}^2a^2H_\sigma^2 - \bar{\rho}^2a^2\phi_\sigma H_\sigma + \sum g_i(S)\delta(t-t_i)\right)ds + \bar{\rho}aH_\sigma dW_s^\perp. \tag{3.11}$$

Integrating equations (3.9) and (3.11) yields the following result.

Theorem 3.5. *The payoff P admits the decomposition*

$$P = \sum_{i=1}^{n} g_i(S_{t_i}) = H(S_t,\sigma_t,t) + \int_t^T \left(S_sH_S + \rho\frac{a}{\sigma}H_\sigma\right)\frac{dS_s}{S_s}$$

$$+\int_t^T \left(\frac{1}{2}\gamma\bar{\rho}^2a^2H_\sigma^2 - \bar{\rho}^2a^2\phi_\sigma H_\sigma\right)ds + \int_t^T \bar{\rho}aH_\sigma dW_s^\perp. \tag{3.12}$$

4. Asymptotic Expansions

In the previous section, we derived the market price of volatility risk for the relative indifference price of an option. It was shown to depend on market parameters and, more importantly, on the derivatives of ϕ and H with respect to σ. In the sequel, we suggest two asymptotic expansions that will let us interpret these derivatives in terms of the Vega of various options. The Vega of an option, defined as the sensitivity of the option price with respect to the volatility parameter σ, is a commonly used risk management tool for option traders. The following analysis therefore provides a theoretical justification for its widespread use among practitioners.

Small γ Approximation. To gain some intuition for the price for investors that are close to risk neutral, we construct a power series expansion for small γ:

$$f^*(S,\sigma,t) = f^{(0)}(S,\sigma,t) + \gamma f^{(1)}(S,\sigma,t) + \dots$$
$$H(S,\sigma,t) = H^{(0)}(S,\sigma,t) + \gamma H^{(1)}(S,\sigma,t) + \dots$$

Inserting these approximations into equations (3.4) and (2.11), we deduce that $f^{(0)}$ solves

$$\begin{cases} f_t^{(0)} + \mathcal{L}^{S,\sigma}f^{(0)} + \bar{\rho}^2a^2\phi_\sigma f_\sigma^{(0)} = 0 \\ f^{(0)}(S,\sigma,T) = g(S) \end{cases}$$

and that $H^{(0)}$ solves

$$\begin{cases} H_t(0) + \mathcal{L}^{S,\sigma}H^{(0)} + \bar{\rho}^2a^2\phi_\sigma H_\sigma^{(0)} = \sum_{i=1}^{n-1} g_i(S)\cdot\delta(t-t_i) \\ H^{(0)}(S,\sigma,t_n) = g_n(S). \end{cases}$$

It then follows from the Feynman-Kac formula that the solutions are given by

$$f^{(0)}(S,\sigma,t) = E_{\tilde{\mathbb{Q}}}[g(S_T)|S_t = S, \sigma_t = \sigma]$$

and

$$H^{(0)}(S,\sigma,t) = E_{\tilde{\mathbb{Q}}}[\sum_{i=1}^{n} g_i(S_{t_i})|S_t = S, \sigma_t = \sigma].$$

Notice that $f^{(0)}(S,\sigma,t)$ and $H^{(0)}(S,\sigma,t)$ are the minimal entropy prices of the claim g and the portfolio P, respectively.

Considering the term of order γ, we deduce that $f^{(1)}$ satisfies

$$\begin{cases} f_t^{(1)} + \mathcal{L}^{S,\sigma} f^{(1)} + \bar{\rho}^2 a^2 \phi_\sigma f_\sigma^{(1)} = \bar{\rho}^2 a^2 H_\sigma^{(0)} f_\sigma^{(0)} \\ f^{(1)}(s,\sigma,T) = 0. \end{cases}$$

Applying the Feynman-Kac formula, we obtain

$$f^{(1)}(S,\sigma,t) = -E_{\tilde{\mathbb{Q}}}\left[\bar{\rho}^2 \int_t^T a^2(\sigma_u,u) H_\sigma^{(0)}(S_u,\sigma_u,u) f_\sigma^{(0)}(S_u,\sigma_u,u)du | S_t = S, \sigma_t = \sigma\right].$$

The above expression indicates the *necessary adjustment* with respect to the minimal entropy price. Since the risk aversion coefficient γ is always positive, if $H_\sigma^{(0)}$ and $f_\sigma^{(0)}$ have the same sign, the trader should quote a price that is *lower* than the minimal entropy price. Intuitively, the marginal option will be perceived by the trader as enhancing the volatility risk of the portfolio, so its price ought to be discounted. On the other hand, if $H_\sigma^{(0)}$ and $f_\sigma^{(0)}$ are opposite in sign, the option will be perceived as *risk-reducing*, and should be bought or sold at a premium with respect to the minimal entropy price.

Small ρ and Slow Volatility Approximation. We now present an expansion that will yield expressions in terms of the partial derivative of the Black and Scholes price with respect to the volatility parameter, known to practitioners as the Black-Scholes Vega.

We will first assume that the correlation is small and replace ρ by $\sqrt{\epsilon}\hat{\rho}$. This approximation is commonly used in Foreign Exchange markets, where periods of high volatility are not necessarily associated with downward or upward moves in the exchange rate. Second, we assume that the volatility has slow scale asymptotics, as motivated by Ilhan, Jonsson and Sircar (2004)(see Sircar and Zariphopoulou (2005) for the fast mean-reversion asymptotics). Formally, we replace b by $\epsilon\hat{b}$ and a by $\sqrt{\epsilon}\hat{a}$. We look for expansions of the form:

$$\begin{aligned} f^*(S,\sigma,t) &= f^{(0)}(S,\sigma,t) + \epsilon f^{(1)}(S,\sigma,t) + \dots \\ H(S,\sigma,t) &= H^{(0)}(S,\sigma,t) + \epsilon H^{(1)}(S,\sigma,t) + \dots \\ \phi(S,\sigma,t) &= \phi^{(0)}(S,\sigma,t) + \epsilon \phi^{(1)}(S,\sigma,t) + \dots \end{aligned}$$

and substitute into equations (3.4), (2.11) and (2.7). We will introduce the operator

$$\mathcal{L}^{(BS)} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial \sigma^2}.$$

It follows that $f^{(0)}$ solves

$$\begin{cases} \mathcal{L}^{(BS)} f^{(0)} = 0 \\ f^{(0)}(S,\sigma,T) = g(S), \end{cases}$$

while $H^{(0)}$ and $\phi^{(0)}$ solve respectively

$$\begin{cases} \mathcal{L}^{(BS)} H^{(0)} = \sum_{i=1}^{n-1} g_i(S) \cdot \delta(t - t_i) \\ H^{(0)}(S,\sigma,t_n) = g_n(S) \end{cases}$$

and

$$\begin{cases} \phi_t^{(0)} = \frac{1}{2}\frac{\mu^2}{\sigma^2} \\ \phi^{(0)}(\sigma, t_{\max}) = 0. \end{cases}$$

Therefore

$$f^{(0)}(S, \sigma, t) = E_{\mathbb{Q}}[g(S_T)|S_t = S]$$

and

$$H^{(0)}(S, \sigma, t) = E_{\mathbb{Q}}[\sum_{i=1}^{n} g_i(S_{t_i})|S_t = S].$$

Notice that $f^{(0)}(S, \sigma, t)$ and $H^{(0)}(S, \sigma, t)$ are the Black and Scholes prices of the claim g and the portfolio P, with volatility *fixed* at the initial value σ, and $\mathbb{Q}$ is the unique martingale measure.

The function $\phi^{(0)}$ is independent of the option to be priced and is given by

$$\phi_t^{(0)} = -(t_{\max} - t)\frac{\mu^2}{2\sigma^2}.$$

The first order term $f^{(1)}$ solves:

$$\begin{cases} \mathcal{L}^{(BS)} f^{(1)} = -\frac{1}{2}\hat{a}^2 f_{\sigma\sigma}^{(0)} - \hat{\rho}\hat{a}\sigma S f_{\sigma S}^{(0)} - (\hat{b} - \hat{\rho}\hat{a}\frac{\mu}{\sigma}) f_\sigma^{(0)} \\ \qquad\qquad -\hat{a}^2 \phi_\sigma^{(0)} f_\sigma^{(0)} + \gamma \hat{a}^2 H_\sigma^{(0)} f_\sigma^{(0)} \\ f^{(1)}(S, \sigma, T) = 0. \end{cases}$$

Applying the Feynman-Kac formula we obtain the first order corrections to the Black and Scholes formula

$$\epsilon f^{(1)}(S, \sigma, t) = f^{\text{volga}} + f^{\text{vanna}} + f^{\text{vega}}$$

where

$$f^{\text{volga}} = E_{\mathbb{Q}}\left[\int_t^T \frac{1}{2} a^2(\sigma, u) f_{\sigma\sigma}^{(0)} du | S_t = S, \sigma_t = \sigma\right], \tag{4.1}$$

$$f^{\text{vanna}} = E_{\mathbb{Q}}\left[\int_t^T \rho a(\sigma, u)\sigma S f_{\sigma S}^{(0)} du | S_t = S, \sigma_t = \sigma\right] \tag{4.2}$$

and

$$\begin{aligned} f^{\text{vega}} = E_{\mathbb{Q}}\Big[\int_t^T \Big(& b(\sigma, u) - \rho a(\sigma, u)\frac{\mu}{\sigma} - a^2(\sigma, u)\phi_\sigma^{(0)} \\ & -\gamma a^2(\sigma, u) H_\sigma^{(0)}\Big) f_\sigma^{(0)} du | S_t = S, \sigma_t = \sigma\Big] \end{aligned} \tag{4.3}$$

The partial derivatives of the Black-Scholes formula with respect to model parameters can be computed explicitly for call and put options. Despite their names, Volga, Vanna and Vega are part of the large family of Black-Scholes sensitivities known as "Greeks".

5. Numerical Results

In this section, we compute the relative indifference prices for European call options, given different scenarios of portfolios held by the trader. In particular, we compute implied volatility surfaces when the trader

(i) holds no options (Figure 1),

(ii) is long or short 6 month at-the-money calls (Figure 2),

(iii) is long or short 18 month at-the-money calls (Figure 3) and

(iv) has a mixed position that is long 6 month and 18 month at-the-money calls and short 12 month at-the-money calls (Figure 4).

We specify a simple SABR model (see Hagan et al. (2002)) for the volatility dynamics by letting $b(\sigma, t) = 0$ and $a(\sigma, t) = \alpha\sigma$. In the numerical simulations below, we assume that $\mu = 0$, $\sigma = 0.1$, $\alpha = 0.5$, $\rho = 0.02$, $\gamma = 0.01$ and $t = 0$. We then use the small ρ and slow volatility approximation described in Section 4 to derive call prices for various maturities and strikes. The maturities T we consider are the 6 month, 9 month, 12 month and 18 month maturities. Note that the strikes K are expressed in terms of the 25 delta (out of the money), 50 delta (at the money) and 75 delta (in the money) European calls, which are the some of the most standard options in FX markets. See Wystup (2002) for a description of the market conventions.

The first step is to compute the Black and Scholes prices for the various options at the initial volatility and stock price, namely

$$f^{(0)}(K, T) = SN(d_1) - KN(d_2),$$

where d_1 and d_2 are the usual expressions. We then discretize the integrals in expressions (4.1), (4.2) and (4.3) to obtain the first order adjustments,

$$f^{\text{volga}}(K, T) = \sum_j (t_{j+1} - t_j) \frac{1}{2} \alpha^2 \sigma^2 E\left[f^{(0)}_{\sigma\sigma}(S_{t_j}, t_j)\right]$$

$$f^{\text{vanna}}(K, T) = \sum_j (t_{j+1} - t_j) \rho\alpha\sigma^2 E\left[S_{t_j} f^{(0)}_{\sigma S}(S_{t_j}, t_j)\right]$$

and

$$f^{\text{vega}}(K, T) = -\sum_j (t_{j+1} - t_j) \gamma\alpha^2\sigma^2 E\left[H^{(0)}_{\sigma}(S_{t_j}, t_j) f^{(0)}_{\sigma}(S_{t_j}, t_j)\right]. \tag{5.1}$$

Interpretation. The Greeks in the first two formulae,

$$f^{(0)}_{\sigma\sigma}(S, t) = S \frac{\sqrt{T - t}}{\sqrt{2\pi}} \exp\left(-\frac{d_1^2}{2}\right) \frac{d_1 d_2}{\sigma}$$

and

$$f^{(0)}_{\sigma S}(S, t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_1^2}{2}\right) \frac{d_2}{\sigma}$$

are the well known industry indices, "Volga" and "Vanna". Volga is viewed by FX traders as an explanation for the symmetric part of the smile while Vanna is viewed as an explanation for the skew-symmetric part (see Wystup (2002)). Indeed, the term f^{volga} is approximately symmetric around the at-the-money strike ($K = S$) and f^{vanna}

is approximately skew-symmetric around the at-the-money strike. If we assume that the trader is not holding any options ($P = 0$) or is risk-neutral ($\gamma = 0$), only the Volga and Vanna corrections to the Black and Scholes price will be necessary and the implied volatility surface of our model will look like Figure 1.

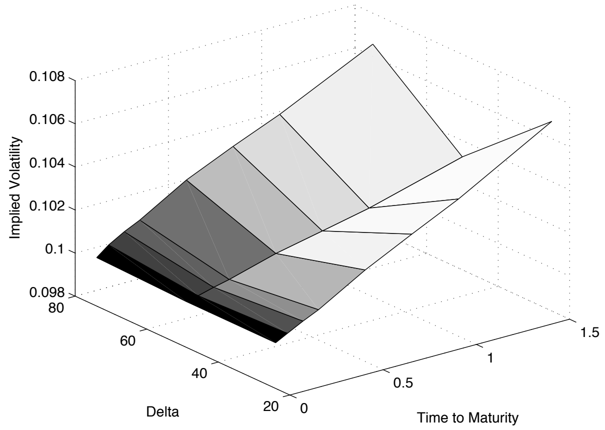

FIGURE 1. The implied volatility surface given by a SABR model with $\sigma = 0.1$, $\alpha = 0.5$ and $\rho = 0.02$

It is a common feature of diffusion models for volatility, that they fail to generate significant smiles for short dated options. In practice, in most foreign exchange markets, short dated options have large smiles, an empirical fact that could be addressed through the modeling of jumps in the spot or volatility process. The main range of applicability that we will consider is therefore the middle term foreign exchange options in the 6 month to 18 month range.

The third correction term f^{vega} will be the main focus of our simulations. It involves the interaction between the Vega of the option

$$f_\sigma^{(0)}(S,t) = S\frac{\sqrt{T-t}}{\sqrt{2\pi}}\exp{(-\frac{d_1^2}{2})}$$

and the Vega of the portfolio

$$H_\sigma^{(0)}(S,t) = \sum_{\{i|t_i \geq t\}} S\frac{\sqrt{t_i-t}}{\sqrt{2\pi}}\exp{(-\frac{(d_1^i)^2}{2})}.$$

Our approximate Vega adjustment formula in equation (5.1) suggests that the trader keep track of the Vega of his portfolio $H_\sigma^{(0)}$ at the future times t_j. The premium he should charge on the Black-Scholes option price $f^{(0)}$ is a weighted average of the expected product of $H_\sigma^{(0)}$ and $f_\sigma^{(0)}$ at these times. In fact, the time intervals during which the two Vegas have the same sign will result in a negative contribution to the option price, while the time intervals during which the two Vegas have different signs will result in a positive contribution to the option price.

For example, if the trader has bought many calls and wants to quote a price on an additional call, the signs of $H_\sigma^{(0)}$ and $f^{(0)}$ will both be positive (since calls have positive Black-Scholes Vegas), and the Vega correction will therefore be negative.

In Figure 2, we illustrate this intuition by comparing the volatility surface without a portfolio (same as in Figure 1) to the scenarios of a trader who is either short or long 6 month at-the-money calls. If the trader has sold 1,500 calls, for instance, the implied volatility surface is shifted upward, with a shift that is more significant for 6 month options than for 18 month options. This indicates that the trader is willing to pay a higher price for marginal calls, since this will tend to reduce the volatility risk of the portfolio book. Of course, if the trader has bought 1500 calls, the intuition is reversed and he will be happy to unload calls at a discount price.

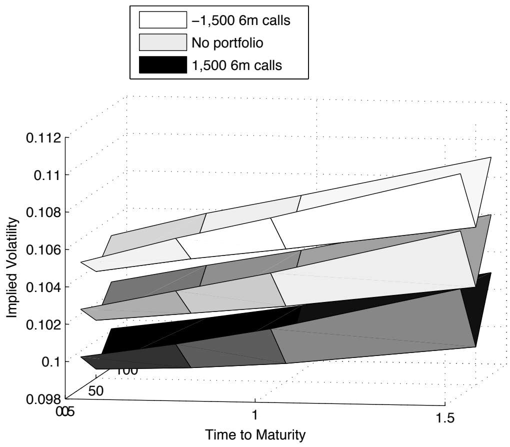

FIGURE 2. The trader has sold or bought 1500 at-the-money 6 month calls

In Figure 3, we repeat the simulation for a trader who is either short or long 18 month at-the-money options. If the trader has sold 500 calls, the whole implied volatility surface is still shifted upwards, but the shift is more pronounced for the 18 month options.

In practice, traders hold complex portfolios made up of short and long positions for various strikes and maturities. The interaction between such a portfolio Vega and the options to be priced can give rise to interesting implied volatility surfaces, as illustrated in Figure 4. Here we compare the implied volatility surface for a trader who has bought 6 month and 18 month at-the-money calls (5,000 units of each) and sold 12 month at-the-money calls (10,000 units). The corresponding volatility structure has a "bump" at the 12 month maturity, indicating that he should feel more reluctant to sell more 12 month calls, but is willing to buy them at a premium to unwind the position.

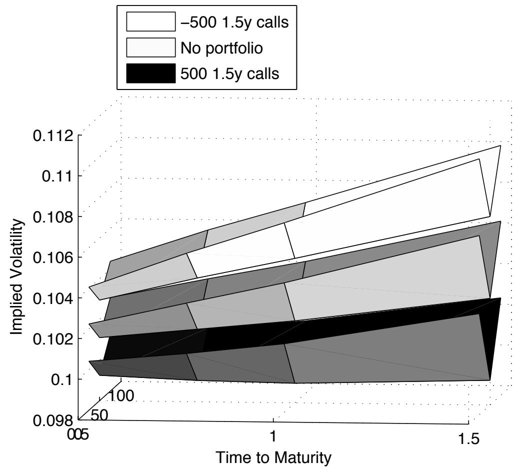

FIGURE 3. The trader has sold or bought 500 at-the-money 18 month calls

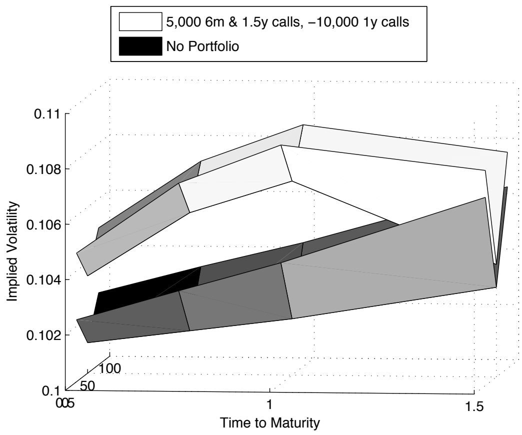

FIGURE 4. Comparing the implied volatility surfaces for a trader with a mixed portfolio to the benchmark scenario with no portfolio

APPENDIX

Proof of Theorem 2.3. For $t \in (t_{n-1}, t_n]$, the value function with the portfolio is the same as the value function with a single option,

$$V^P(x, \sigma, S, t, t_n) = V^{g_n}(x, \sigma, S, t, t_n).$$

It follows that the indifference price of the portfolio coincides with the price of the option given by Theorem 2.1. Therefore $H(S, \sigma, t)$ for $t \in (t_{n-1}, t_n]$ is the unique $C^{2,2,1}$ bounded solution of equation (2.6) with terminal condition $H(S, \sigma, t_n) = g_n(S)$.

We proceed by induction on the times t_i. First assume that for $t \in (t_i, t_n]$, $H(S, \sigma, t)$ is the unique $C^{2,2,1}$ bounded solution of equation (2.11). At time t_i, it follows from the definition of V^P that

$$V^P(x,\sigma,S,t_i,t_n) = V^P(x+g_i(S),\sigma,S,t_i^+,t_n).$$

Using the indifference relation we obtain

$$V(x+H(S,\sigma,t_i),\sigma,S,t_i,t_n) = V(x+g_i(S)+H(S,\sigma,t_i+),\sigma,S,t_i^+,t_n)$$

and the continuity of V yields the pasting condition

$$H(S,\sigma,t_i) = g_i(S) + H(S,\sigma,t_i^+).$$

Once again, applying Theorem 2.1. for $t \in (t_{i-1}, t_i]$ with the (bounded and smooth) terminal condition $H(S,\sigma,t_i)$, we obtain that $H(S,\sigma,t)$ is the unique $C^{2,2,1}$ bounded solution of equation (2.11) for $t \in (t_{i-1}, t_n]$. The theorem follows by induction.

□

Proof of Theorem 3.2. Recall that the portfolio indifference price for P solves

$$\left\{ \begin{array}{l} H_t + \mathcal{L}^{S,\sigma} H + \bar{\rho}^2 a^2 \phi_\sigma H_\sigma - \frac{1}{2}\gamma\bar{\rho}^2 a^2 H_\sigma^2 = \sum_{i=1}^{n-1} g_i(S)\cdot\delta(t-t_i) \\ H(S,\sigma,t_n) = g_n(S). \end{array} \right. \tag{5.2}$$

The portfolio indifference price for $P+g$, which we denote by

$$\nu_t(P+g) = H^*(S,\sigma,t),$$

solves

$$\left\{ \begin{array}{l} H_t^* + \mathcal{L}^{S,\sigma} H^* + \bar{\rho}^2 a^2 \phi_\sigma H_\sigma^* - \frac{1}{2}\gamma\bar{\rho}^2 a^2 (H_\sigma^*)^2 \\ = \sum_{i=1}^{n-1} g_i(S)\cdot\delta(t-t_i) + g(S)\cdot\delta(t-T) \\ H^*(S,\sigma,t_n) = g_n(S). \end{array} \right. \tag{5.3}$$

Since $f = H^* - H$, we may subtract equations (5.2) from (5.3) to obtain the desired result. □

Proof of Theorem 3.4. We present the formal derivation for the optimal investment with and without the portfolio of options. Since all the PDE involved admit unique classical solutions, we may apply classical verification results (see Theorem IV.3.1. in Fleming and Soner (1993)) to ensure the optimality of our solutions. The function $V(x,\sigma,t)$ solves the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\left\{ \begin{array}{l} V_t + \max_\pi[\frac{1}{2}\sigma^2\pi^2 V_{xx} + \rho\sigma\pi a V_{x\sigma} + \mu\pi V_x] + \frac{1}{2}a^2 V_{\sigma\sigma} + bV_\sigma = 0 \\ V(x,\sigma,t_n) = -e^{-\gamma x} \end{array} \right.$$

After inserting the maximal value of the control

$$\pi^0 = -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} - \rho\frac{a}{\sigma}\frac{V_{x\sigma}}{V_{xx}} \tag{5.4}$$

and substituting

$$V(x,\sigma,t) = -\exp(-\gamma x)\exp(\gamma\phi(\sigma,t)) \tag{5.5}$$

we find, after some tedious calculations, that ϕ solves

$$\left\{ \begin{array}{l} \phi_t + \mathcal{L}^{(\sigma)}\phi + \frac{1}{2}\bar{\rho}^2 a^2 \phi_\sigma^2 = \frac{\mu^2}{2\sigma^2} \\ \phi(\sigma,t_n) = 0. \end{array} \right. \tag{5.6}$$

Substituting (5.5) into (5.4) yields

$$\pi^0 = \frac{\mu}{\gamma\sigma^2} + \rho\frac{a}{\sigma}\phi_\sigma. \tag{5.7}$$

The function $V^P(x,\sigma,S,t)$ solves the following HJB equation

$$\begin{cases} V_t + \max_\pi[\frac{1}{2}\sigma^2\pi^2 V_{xx} + \rho\sigma\pi a V_{x\sigma} + \sigma^2 s\pi V_{xS} + \mu\pi V_x] \\ +\frac{1}{2}a^2 V_{\sigma\sigma} + \rho\sigma S a V_{\sigma S} + \frac{1}{2}\sigma^2 S^2 V_{SS} + bV_\sigma + \mu S V_S \\ = -\exp(-\gamma x)\exp(-\gamma\sum_{i=1}^{n-1} g_i(S)) \cdot \delta(t-t_i) \\ V(x,S,\sigma,t_n) = -e^{-\gamma x - \gamma g_n(S)} \end{cases}$$

where $\delta()$ is the Dirac delta function. Inserting the maximal value of the control

$$\pi^P = -\frac{\mu}{\sigma^2}\frac{V_x^P}{V_{xx}^P} - \rho\frac{a}{\sigma}\frac{V_{x\sigma}^P}{V_{xx}^P} - S\frac{V_{xS}^P}{V_{xx}^P} \tag{5.8}$$

with the ansatz

$$V^P(x,\sigma,S,t) = -e^{-\gamma x + \gamma(\phi(\sigma,t) - H(S,\sigma,t))} \tag{5.9}$$

and using equation (5.6) yields, after some very tedious calculations that H solves,

$$\begin{cases} H_t + \mathcal{L}^{(\sigma,S)}H + \bar{\rho}^2 a^2 \phi_\sigma H_\sigma - \frac{1}{2}\gamma\bar{\rho}^2 a^2 H_\sigma^2 = \sum_{i=1}^{n-1} g_i(S)\cdot\delta(t-t_i) \\ H(s,\sigma,t_n) = g_n(S). \end{cases} \tag{5.10}$$

Notice that $H(S,\sigma,t)$ is the indifference price of the portfolio, since

$$V(x + H(S,\sigma,t),\sigma,t,t_n) = V^P(x,\sigma,S,t,t_n).$$

Substituting (5.9) in (5.8), we obtain the optimal investment policy of the trader who is holding a book of options,

$$\pi_s^P = \frac{\mu}{\gamma\sigma^2} + \rho\frac{a}{\sigma}(\phi_\sigma - H_\sigma) - SH_S. \tag{5.11}$$

Subtracting equation (5.7) from (5.11) yields the desired result. □

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