OPTIMAL INVESTMENTS IN THE PRESENCE OF UNHEDGEABLE RISKS AND UNDER CARA PREFERENCES

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Abstract. We provide a structural charaterization result for the non-myopic optimal portfolio of a CARA agent who invests in an incomplete market environment. The excess risky demand turns out to be the indifference risk monitoring strategy of an emerging claim written on the traded asset's Sharpe ratio and the risk tolerance of the investor. Sensitivity results are provided as well as explicit formulae for indifference prices of path-dependent claims written on non-traded assets.

Key words. Stochastic Sharpe ratio, indifference prices, non-myopic investment, excess risky demand

AMS(MOS) subject classifications. 93E20, 60G40, 60J75

1. Introduction.

This paper analyzes the optimal investment decisions of a CARA agent in an incomplete market environment. The agent trades between a riskless bond and a risky stock whose price is a diffusion with dynamics affected by a correlated stochastic factor. Risk preferences are exponential and the agent's objective is to maximize his/her expected utility of terminal wealth. In this setting, the optimal investment consists of two components, the so-called *myopic* and *non-myopic* portfolio. The myopic policy is the one that the investor would follow ignoring what would happen beyond the immediate next period. It does not depend on the distribution of asset returns over future revision intervals. The non-myopic investment emerges from the stochasticity of the opportunity set and reflects how the investor reacts to risks that cannot be eliminated. It is also known as the *excess risky demand* and is the main focus of our study.

Maximal expected utility problems have been widely studied for models of general asset dynamics and arbitrary preferences. By far, the most popular method is based on duality arguments. Variational methods have been also applied for certain diffusion market dynamics and risk preferences, and explicit solutions have been produced for special cases.

In all these works, however, the emphasis is on the value function and not on the optimal policies. The latter can be analyzed through martin-

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This work was presented at the Mathematical Finance for Young Researchers Conference at Humboldt University of Berlin (January 2004), the Workshop on Management Specification Issues in Finance, IMA, Minneapolis (April 2004) and at the 3rd World Congress of the Bachelier Finance Society at Chicago (July 2004). The authors would like to thank the participants for fruitful comments and suggestions.

gale representation results emerging in the analysis of the dual problem (Kramkov and Schachermayer (1999)). In diffusion settings, they can be produced from the primary problem via the first order conditions in the associated Hamilton-Jacobi-Bellman equation; see, among others, Kim and Omberg (1996), Wachter (2002) as well as Campbell and Viceira (1999) where policies are constructed with the so called log-linear approximation (see, also, Campbell et al. (2004) and Chacko and Viceira (1999)). Generally speaking, the excess risky demand has been associated with hedging of some endogenous risk. In complete markets, this can been fully justified. The dual pde is linear and an emerging pseudo-derivative provides, through its associated classical arbitrage free hedging, a natural link between optimal investment and risk replication. In incomplete markets however, such a natural bridge has not been, in our view, fully developed. The notion of hedging is non transparent, and the connection between optimal behavior and risk monitoring has not been satisfactorily built.

Herein, we concentrate on a specific class of risk preferences and we propose an approach that seems to provide a coherent link between optimal portfolios and risk monitoring strategies. Three ingredients are needed. The supporting claim, the valuation approach and the associated risk management. We show that the relevant claim is written on the stochastic factor with payoff depending on the Sharpe ratio of the traded stock and the risk tolerance of the investor. It is valuated by the *indifference* method. Its risk monitoring strategies, that ultimately yield the excess risky demand for the original investment model, are determined through the associated portfolios, with and without the claim. For this, two important notions are introduced, namely, the residual optimal wealth and the residual risk. In complete markets, the residual risk vanishes and the residual optimal wealth coincides with the risk replicating portfolio. In the presence of unhedgeable risks, however, these quantities carry a lot of information for the quantification of the accumulated risk and the size of the replicable payoff component. Through them, we provide a meaningful payoff decomposition and, in turn, a natural link between optimal investment and *indifference hedging*. Besides the structural analysis and characterization of the optimal non-myopic portfolios, we discuss their behavior in terms of various market inputs. We subsequently study another class of incomplete models with lognormal stock dynamics and stochastic preferences. We find conditions on the market coefficients so that these models have the same dynamic utility as the ones studied earlier. This enable us to construct indifference prices of path-dependent claims written on non-traded assets. These findings can be readily used for the valuation of labor income streams (see Henderson (2004a) for a special case) and proprietorship contracts (Zariphopoulou (2004)).

The paper is organized as follows. In section 2, we introduce the optimal investment model and present preliminary results for its value function. In section 3, we revisit the notion of indifference price and we derive the valuation functionals for path-dependent claims written on the stochastic factor. The main results are given in Section 4 where the supporting claim is introduced and priced. We present the payoff decomposition and we establish the relation between the emerging risk monitoring component and the original non-myopic portfolio. Sensitivity analysis is also provided. In section 5, we analyze an alternative incomplete model and we construct indifference prices for path-dependent claims written on non-traded assets. Conclusions and further directions are discussed in Section 6.

2. The optimal investment model and preliminary results.

i) The model and the associated HJB equation

We consider an optimal investment model for a single agent who manages his/her portfolio by investing in a stock and a riskless bond. The price of the stock S solves

$$dS_s = \mu(Y_s, s)S_s ds + \sigma(Y_s, s)S_s dW_s^1 \tag{2.1}$$

with $S_t = S \geq 0$. The process Y will be referred to as the *stochastic factor* and it is assumed to satisfy

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s \tag{2.2}$$

with $Y_t = y \in \mathcal{R}$. The processes W^1 and W are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ with $\mathcal{F}_s$ being the augmented σ-algebra and $\rho \in (-1, 1)$ the correlation coefficient. We then have that $W_s = \rho W_s^1 + \sqrt{1-\rho^2} W_s^{1,\perp}$ with $W_s^{1,\perp}$ being a Brownian motion orthogonal to W_s^1 under $\mathbb{P}$. Assumptions on the drift and diffusion coefficients μ, σ, a and b are such that equations (2.1) and (2.2) have a unique strong solution satisfying $S_s \geq 0$ *a.e.* for $s \geq t$. The bond is assumed to offer zero interest rate. The case of (deterministic) non-zero interest rate may be handled by straightforward scaling arguments and is not discussed.

The investor starts at time $t \in [0, T)$ with initial wealth $x \in \mathcal{R}$. His/her current wealth X_s, $t \leq s \leq T$, satisfies the budget constraint $X_s = \pi_s^0 + \pi_s$ where π_s^0 and π_s are, respectively, the amounts allocated in the bond and the stock accounts. No trading constraints, intermediate consumption nor exogenous stream of funds are allowed. Direct calculations involving the dynamics in the above equations yield the evolution of the wealth process

$$dX_s = \mu(Y_s, s)\pi_s ds + \sigma(Y_s, s)\pi_s dW_s^1 \tag{2.3}$$

with $X_t = x \in \mathcal{R}$.

The investor is endowed with *Constant Absolute Risk Aversion* (CARA) utility

$$U(x) = -e^{-\gamma x}, \quad x \in \mathcal{R} \tag{2.4}$$

and has value function

$$V(x,y,t;\pi) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(U(X_T) \,|X_t = x, Y_t = y \right). \tag{2.5}$$

The processes Y and X are given, respectively, by (2.2) and (2.3) and the risk aversion coefficient satisfies $\gamma > 0$. The set $\mathcal{A}$ of admissible portfolio policies is defined as $\mathcal{A} = \{\pi : \pi_s$ is $\mathcal{F}_s-$ measurable and $E_{\mathbb{P}} \int_t^T \sigma^2(Y_s,s)\pi_s^2 ds$ $< \infty\}$.

We denote the *Sharpe ratio* process of the traded asset by

$$\lambda_s = \lambda(Y_s,s) = \frac{\mu(Y_s,s)}{\sigma(Y_s,s)}. \tag{2.6}$$

The following assumption on the coefficients will be standing throughout:

Assumption 1: The market coefficients λ, a and b are assumed to be $C^{1,2}([0,T]\times\mathcal{R})$ functions that satisfy, uniformly in t, $|f(y,t)| \leq C\,(1+|y|)$, for $f = \lambda, a$ and b. There also exists $\varepsilon > 0$ such that $\sigma\,(y,t) > \varepsilon$, for $(y,t) \in (\mathcal{R}\times[0,T])$.

For the rest of the presentation we suppress the arguments of the various coefficients and we reinstate them whenever needed. We also introduce the differential operators

$$\mathcal{L} = \frac{1}{2}a^2\frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial y}, \tag{2.7}$$

$$\mathcal{L}^{mm} = \frac{1}{2}a^2\frac{\partial^2}{\partial y^2} + (b-\rho\lambda a)\frac{\partial}{\partial y} \tag{2.8}$$

and the domain $\mathcal{D} = \mathcal{R}\times\mathcal{R}\times[0,T]$.

Proposition 2.1. *The value function is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation*

$$V_t + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi(\rho\sigma a V_{xy} + \mu V_x)\right) + \mathcal{L}V = 0, \tag{2.9}$$

and $V(x,y,T) = -e^{-\gamma x}$ with $\mathcal{L}$ as in (2.7) and $(x,y,t) \in \mathcal{D}$. Moreover, it is the unique such solution in the class of functions that are concave and nondecreasing in x, and, for fixed (x,t), bounded in y.

The proof is omitted since it follows along similar arguments used in Theorems 4.1 and 4.2 in Duffie and Zariphopoulou (1993).

ii) Representations of the value function and related measures

The stochastic control problem (2.5) has been extensively analyzed in a much broader model setting and under minimal assumptions on the price processes (see, among others, Rouge and El Karoui (2000), Delbaen et. al. (2002), and Kabanov and Stricker (2002)). The analysis was based on duality methods which gave rise to reduced optimization problems involving entropic criteria.

A measure that emerges naturally in this approach is the *minimal relative entropy* measure, denoted herein by $\mathbb{Q}^{me}$. It minimizes the relative entropy

$$\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P}) = \min_{Q \in \mathcal{Q}^e} E_{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}} \ln \frac{dQ}{d\mathbb{P}}\right),$$

among all martingale measures Q that are equivalent to the historical measure $\mathbb{P}$. Henceforth, we will denote by $\mathcal{Q}^e$ the set of all such measures. We refer the reader to the papers of Frittelli (2000) (see, also, Grandits and Rheinlander (2002) and Arai (2001)) for an extensive study of the minimal relative entropy measure in optimization problems related to asset and derivative valuation.

The following result was shown in Delbaen et al. (2002) (see also Rouge and El Karoui (2000)).

Proposition 2.2. *Let $\mathbb{Q}^{me}$ be the minimal relative entropy measure. Let also λ_s and $\lambda_s^{\perp}$ be, respectively, the Sharpe ratio (2.6) and the process appearing in the representation of the relative density*

$$\frac{d\mathbb{Q}^{me}}{d\mathbb{P}} = \exp\left(\int_0^T -\lambda_s dW_s^1 - \int_0^T \lambda_s^{\perp} dW_s^{1,\perp} - \frac{1}{2}\int_0^T \left(\lambda_s^2 + \left(\lambda_s^{\perp}\right)^2\right) ds\right). \tag{2.10}$$

The value function V is given, for $(x,y,t) \in \mathcal{D}$, by

$$V(x,y,t) = -e^{-\gamma x - v(y,t)} = U\left(x + \frac{1}{\gamma} v(y,t)\right) \tag{2.11}$$

with

$$v(y,t) = E_{\mathbb{Q}^{me}}\left(\int_t^T \frac{1}{2}\left(\lambda(Y_s,s)^2 + \lambda^{\perp}(Y_s,s)^2\right) ds \,|Y_t = y\right). \tag{2.12}$$

For the model at hand, we may derive an alternative representation for the value function. It is analogous to the one introduced in Zariphopoulou (2001) (see, also, Tehranchi (2004)) for an investment model under Constant Relative Risk Aversion (CRRA) preferences. There, the value function was constructed in terms of a power (distortion) transformation of a solution to a linear pde. When preferences are exponential, a similar structure also appears as is exhibited in formula (2.18).

The interesting ingredient in this representation is the emerging measure, denoted by $\mathbb{Q}^{mm}$. This turns out to be the *minimal martingale measure*, introduced by Foellmer and Schweizer (1991) (see also Schweizer (1995) and (1999)). It is an equivalent martingale measure that minimizes

$$\mathcal{H}^0(\mathbb{Q}^{mm}|\mathbb{P}) = \min_{Q\in\mathcal{Q}^e} E_{\mathbb{P}}\left(-\ln\frac{dQ}{d\mathbb{P}}\right). \tag{2.13}$$

Direct calculations show that it is given by

$$\mathbb{Q}^{mm}(A) = E_{\mathbb{P}}\left(e^{-\int_0^T \lambda_s dW_s^1 - \int_0^T \frac{1}{2}\lambda_s^2 ds} I_A\right), \quad A \in \mathcal{F}_T, \tag{2.14}$$

and that its restriction on $\mathcal{F}_T^W$ satisfies

$$\mathbb{Q}^{mm}(A) = E_{\mathbb{P}}\left(e^{-\int_0^T \rho\lambda_s dW_s - \int_0^T \frac{1}{2}\rho^2\lambda_s^2 ds} I_A\right), \quad A \in \mathcal{F}_T^W. \tag{2.15}$$

Proposition 2.3. *Let $\lambda_s = \lambda(Y_s, s)$ be the Sharpe ratio process (2.6), $\rho \in (-1, 1)$ the correlation between the stock and the stochastic factor, and $\mathcal{L}^{mm}$ the operator defined in (2.8). Let u be the solution of the terminal value problem*

$$u_t + \mathcal{L}^{mm}u = \frac{1}{2}(1-\rho^2)\lambda^2(y,t)u \tag{2.16}$$

with $u(y, T) = 1$ for $(y, t) \in \mathcal{R}\times[0, T]$. Then

$$u(y,t) = E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds}|Y_t = y\right) \tag{2.17}$$

and the value function V is given, for $(x, y, t) \in \mathcal{D}$, by

$$V(x,y,t) = -e^{-\gamma x}u(y,t)^{1/(1-\rho^2)}. \tag{2.18}$$

Proof. We first construct a candidate solution, say $\hat{V}$, to the HJB equation (2.9). From the scaling properties of the value function, we easily deduce that $\hat{V}(x, y, t) = -e^{-\gamma x}F(y, t)$. Further calculations show that F can be represented as $F = u^{1/(1-\rho^2)}$ with u solving (2.16). Using (2.15), (2.16), the form of $\mathcal{L}^{mm}$ and the Feynman-Kac formula we deduce (2.17).

The identification of the candidate $\hat{V}$ with the value function follows directly from its growth and regularity properties and well known verification results (see Pham (2002) and Zariphopoulou (2003)). □

From the two alternative value function representations (2.11) and (2.18), we may deduce an interesting relation between the minimal relative entropy measure, the minimal martingale measure and the minimal relative entropy. The equality (2.19) below shows that the latter can be expressed in a *linear* and a *nonlinear* fashion.

The linear functional is an expectation, under the minimal entropy measure, of the aggregate (squares of) market prices of traded and non-traded risks. The nonlinear functional has *certainty equivalent* characteristics and the involved measure is the minimal martingale one. We will revisit pricing functionals of this form in the next section when we consider the so-called *indifference prices*. These prices will help us understand the nature of the optimal investment decisions of the CARA agent.

COROLLARY 2.1. *In the incomplete market model that consists of the stock S and the stochastic factor Y, solving (2.1) and (2.2), the minimal relative entropy $\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P}\,)$ satisfies*

$$\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P}\,) = E_{\mathbb{Q}^{me}}\left(\int_0^T \frac{1}{2}{\lambda_s}^2 + \left(\lambda_s^{\perp}\right)^2 ds\right) \tag{2.19}$$

$$= -\frac{1}{1-\rho^2}\ln E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\int_0^T \frac{1}{2}\lambda_s^2 ds}\right)$$

with λ_s and $\lambda_s^{\perp}$ as in (2.6) and (2.10).

We remark that the above nonlinear functional is not a naive extension of the static actuarial certainty equivalent valuation rule. In fact, for the model at hand, a direct dynamic analogue would have been of the form $-\ln E_{\mathbb{P}}(e^{-(\cdot)})$. But (2.19) shows that neither the involved measure nor the valuation functional are in accordance with the traditional insurance static rule.

iii) The complete market case and the emerging path-dependent claim

Formulae (2.11) and (2.18) expose the invariance of the value function with respect to the stock's Sharpe ratio. They also give us some insights on how model incompleteness is compiled. The *path-dependent* term in (2.11) (see also (2.18)) shows that, even though the stock dynamics are affected *locally* by the stochastic factor, the effects of the latter appear on the value function in an *aggregated* form. This observation will help us in two directions. With regards to the optimal investments, it will lead us to the specification of the supporting claim whose risky monitoring strategies will produce the non-myopic demand for the risky asset. In a broader perspective, it will highlight how risk preferences and model incompleteness are interconnected. The specification of the optimal policies will be presented first while the model specification issue is discussed in Section 5.

To gain some intuition on how the above results may be used on the construction and analysis of the optimal investment rules, we first consider the *complete* market case. For this, we may conveniently think of the stock price having nonlinear stock dynamics. This model was first analyzed by Merton (1990) and, subsequently, by many others via either variational or duality arguments.

The solution (2.22) is directly computed from (2.11). It can be also obtained from (2.18) by passing to the limit as $\rho^2 \to 1$. In this case, $(1-\rho^2)^{-1} \to \infty$ and scaling arguments show that the power transformation of (2.18) 'converges' to the exponential transformation (2.11). The rigorous results rely on the stability properties of viscosity solutions and we refer the reader to Musiela and Zariphopoulou ((2001), Theorem 2.3) for the relevant technical arguments.

We note that because the market is complete, the minimal relative entropy measure and the minimal martingale one coincide, and equal to the unique nested risk neutral measure.

PROPOSITION 2.4. *Assume that the stock price S solves*

$$dS_s = \mu(S_s, s)S_s ds + \sigma(S_s, s)S_s dW_s^1. \tag{2.20}$$

Let h be the solution of the terminal value problem

$$h_t + \frac{1}{2}\sigma^2(S,t)S^2 h_{SS} + \frac{1}{2}\lambda^2(S,t) = 0 \tag{2.21}$$

with $h(S,T) = 0$ and $(S,t) \in \mathcal{R}^+ \times [0,T]$. Then

$$h(S,t) = E_{\mathbb{P}^*}\left(\int_t^T \frac{1}{2}\lambda^2(S_s,s)ds \,|S_t = S\right)$$

with $\mathbb{P}^*$ being the risk neutral measure.

The value function V is given by

$$V(x,S,t) = -e^{-\gamma x - h(S,t)} = U\left(x + \frac{1}{\gamma}h(S,t)\right) \tag{2.22}$$

where U is the utility function as in (2.4).

PROPOSITION 2.5. *The optimal investment process π_s^* is given by*

$$\pi_s^* = \pi_s^m + H_s \tag{2.23}$$

where

$$\pi_s^m = \pi^m(S_s, s) \quad \textit{and} \quad H_s = H(S_s, s)$$

with S solving (2.1),

$$\pi^m(S,t) = \frac{1}{\gamma}\frac{\mu(S,t)}{\sigma^2(S,t)} \tag{2.24}$$

and

$$H(S,t) = -\frac{1}{\gamma} S h_S(S,t). \tag{2.25}$$

The proof is a direct adaptation of the one provided in Zariphopoulou (Proposition 2.1, 1999) and it is therefore omitted.

To interpret the above optimal policies, let us first consider a path-dependent claim, introduced at time t and maturing at T, with payoff

$$\Lambda(S_s; t \le s \le T) = \int_t^T -\frac{1}{2}\frac{\lambda^2(S_s,s)}{\gamma} ds. \tag{2.26}$$

It easily follows that its arbitrage-free price process, say $C(S_s,s)$, satisfies $C(S_s,s) = -\frac{1}{\gamma}h(S_s,s)$, $t \le s \le T$ with h solving (2.21). Its risk replication strategy, α_s, is given by

$$\alpha_s = S_s C_S(S_s,s) = -\frac{1}{\gamma} S_s h_S(S_s,s). \tag{2.27}$$

Comparing the above with (2.25) yields

$$H_s = \alpha_s. \tag{2.28}$$

The above analysis shows how the optimal investment strategy of the CARA agent is structured. The *myopic* component, π_s^m, is the amount that the investor would invest in stock if its Sharpe ratio were constant for the next time period. The second term, H_s, is the *non-myopic* optimal investment and equality (2.27) shows that it can be viewed as a risk replicating strategy of the supporting claim Λ.

The advantages of this decomposition are obvious. It provides an intuitive and natural way of bridging the notions of *investment optimality* and *risk replication*. It facilitates the qualitative analysis of optimal strategies by bringing in the well developed technology of 'greeks'. These desirable features strongly motivate us to look for a similar structure even when the market becomes incomplete. Note however that three issues arise, namely, the *identification* of the supporting claim, and the appropriate notion of its *valuation* and *risk replication*. Neither question has an apparent answer due to the inherent nonlinearities and the essential incompleteness of the model.

3. Indifference valuation of path-dependent claims. In this section, we provide some auxiliary results on indifference prices of path-dependent claims. These findings will be used in the next section when we study the structure of the optimal investment strategies of the CARA agent.

The indifference valuation approach has recently gained considerable ground in the pricing and risk quantification in incomplete markets. It is based on optimality of investments that become available to the investors with and without employing (buying or writing) the claim at hand. It produces the so called *indifference price* that represents the payment (compensation) that the buyer (writer) needs to pay (receive) at inscription so that his/her dynamic utility remains unchanged through the life time of the derivative. In most cases so far, the individual preferences have been taken to be of exponential type. Under this assumption, indifference prices have been widely analyzed in general market settings and for arbitrary payoffs (see, among others, Rouge and El Karoui (2000), Delbaen et al. (2002), Bielecki et al (2004)). Duality techniques yield the indifference prices as solutions to optimization problems with entropic penalty terms or as solutions to BSDES.

In Markovian settings, indifference prices turn out to solve *quasilinear* equations with quadratic gradient nonlinearities. In this direction, the analysis has been primarily centered around risks generated by non-traded assets that are correlated with the underlying price process but do not affect its dynamics. In addition, only European and American type claims have been analyzed (see, among others, Musiela and Zariphopoulou (2001, 2004a), and Henderson (2002)). European claims written on a correlated factor that models stochastic volatility have been studied by Sircar and Zariphopoulou (2004) (see, also, Grasselli and Hurd (2004)).

We now consider a claim C, introduced at t, and offering payoff

$$C(Y_s; t \leq s \leq T) = \int_t^T c_1(Y_s, s)ds + c_2(Y_T) \tag{3.1}$$

at the end of the trading horizon T. No cashflows occur in $[t, T)$.

Assumption 2: The coefficients c_i, $i = 1, 2$ are $C^{1,2}\left([0,T] \times \mathcal{R}\right)$ functions, with $|c_i\,(y,t)| \leq C\,(1 + |y|)$ and such that $\sup_{\mathcal{Q}_e} E_Q\left(e^{|C(Y_s; t \leq s \leq T)|}\right) < \infty$.

For the applications at hand, the indifference price to be considered is the one of the *buyer*. His/her value function V^C is defined similarly to (2.5) but with a modified utility payoff reflecting the compensation at expiration, namely,

$$V^C(x, y, t) = \sup_{\mathcal{A}} E_{\mathbb{P}}(-e^{-\gamma(X_T + C(Y_s; t \leq s \leq T))}\,|X_t = x, Y_t = y\,). \tag{3.2}$$

We stress that the compensation C is acquired by the buyer *at the end* of the trading horizon. This precludes trading of intermediate claim proceeds before expiration.

Definition 3.1. *The indifference price of the claim $C(Y_s;t\leq s\leq T)$ is defined as the amount $\nu_t(C)$ for which the value functions V^C and V^0, defined in (3.2) and corresponding, respectively to the claims $C(Y_s;t\leq s\leq T)$ and 0 coincide. Namely, $\nu_t(C)$ is the amount which satisfies*

$$V^0(x,y,t)=V^C(x-\nu_t(C),y,t) \tag{3.3}$$

for all states $(x,y,t)\in\mathcal{D}$.

We note that due to the specific functional form of the path-dependent payoff, V^C can be still written as a function of only two spatial variables.

Definition 3.2. *Let $Q\in\mathcal{Q}^e$ and $Z(Y)=Z(Y_s;t\leq s\leq T)$ where Y solves (2.2), and Z satisfies $\sup_{\mathcal{Q}_e}E_Q\left(e^{|Z|}\right)<\infty$. The nonlinear pricing functional $\mathcal{E}_Q$ is defined by*

$$\mathcal{E}_Q(Z(Y)\,|Y_t=y)=-\frac{1}{\gamma(1-\rho^2)}\ln E_Q\left(e^{-\gamma(1-\rho^2)Z(Y))}\,|Y_t=y\right). \tag{3.4}$$

Recalling identity (2.19) in Corollary 2.1, we may rewrite $\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P})$ as follows.

Corollary 3.1. *The total relative entropy $\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P})$ satisfies*

$$\mathcal{H}(\mathbb{Q}^{me}\,|\mathbb{P})=\mathcal{E}_{\mathbb{Q}^{mm}}\left(\int_0^T\frac{1}{2}\lambda_s^2ds\right)$$

where $\mathcal{E}_Q$ is defined in (3.4), applied for $Q=\mathbb{Q}^{mm}$ and $\gamma=1$.

Next we construct the indifference price of the claim. A *nonlinear* price representation appears and two alternative expressions are presented. One is with regards to the minimal martingale measure (cf. (3.5)) while the other involves the minimal relative entropy measure (cf. (3.14)). For a detailed discussion related to the emerging nonlinear pricing rules we refer the reader to Musiela and Zariphopoulou (2001 and 2004a) and Henderson (2002).

Theorem 3.1. *Let $\mathbb{Q}^{mm}$ be the minimal martingale measure introduced in (2.13) and $\mathcal{E}_Q$ the nonlinear pricing functional (3.4) with $Q=\mathbb{Q}^{mm}$. The buyer's indifference price $\nu_t(C)$ of the path-dependent claim $C(Y_s;t\leq s\leq T)$ is given by*

$$\nu_t(C)=\mathcal{E}_{\mathbb{Q}^{mm}}\left(C(Y_s;t\leq s\leq T)+\int_t^T\frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}ds\,|Y_t=y\right) \tag{3.5}$$

$$-\mathcal{E}_{\mathbb{Q}^{mm}}\left(\int_t^T\frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}ds\,|Y_t=y\right).$$

Proof. We first compute the value functions V^C and V^0. Direct adaptation of the arguments used in the proof of Proposition 2.1 yield that V^C is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

$$V_t^C + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx}^C + \pi(\rho\sigma a V_{xy}^C + \mu V_x^C)\right) + \mathcal{L}V^C = \gamma c_1(y,t)V^C$$

with

$$V^C(x,y,T) = -e^{-\gamma(x+c_2(y))}.$$

We then seek a candidate solution denoted, by a slight abuse of notation, by V^C. It is taken to be of the form $V^C = -e^{-\gamma x}F(y,t)$ with $F(y,t) = u^C(y,t)^{1/(1-\rho^2)}$ and u^C appropriately chosen. The function u^C turns out to be the solution of the linear equation

$$u_t^C + \mathcal{L}^{mm}u^C = (1-\rho^2)\left(\gamma c_1(y,t) + \frac{1}{2}\lambda^2\,(y,t)\right)u^C \tag{3.6}$$

with

$$u^C(y,T) = e^{-\gamma(1-\rho^2)c_2(y)}$$

where $\mathcal{L}^{mm}$ is the operator in (2.8). We easily deduce that the candidate solution coincides with the value function. The Feynman-Kac formula yields

$$V^C(x,y,t) = -e^{-\gamma x}u^C(y,t)^{1/(1-\rho^2)}$$

$$= -e^{-\gamma x}\;\left(E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\left(\gamma C(Y_T)+\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds\right)}\,|Y_t = y\right)\right)^{1/(1-\rho^2)}. \tag{3.7}$$

We recall that for $C \equiv 0$,

$$V^0(x,y,t) = -e^{-\gamma x}u\,(y,t)^{1/(1-\rho^2)}$$

$$= -e^{-\gamma x}\;\left(E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds}\,|Y_t = y\right)\right)^{1/(1-\rho^2)}. \tag{3.8}$$

Combining the above equalities with the definition of the indifference price (3.3), implies

$$-e^{-\gamma x}\left(E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds}\,|Y_t = y\right)\right)^{1/(1-\rho^2)}$$

$$= -e^{-\gamma(x-\nu_t(C))}\;\left(E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\left(\gamma C(Y_T)+\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds\right)}\,|Y_t = y\right)\right)^{1/(1-\rho^2)}.$$

In turn,

$$\nu_t\left(C\right) = -\frac{1}{\gamma\left(1-\rho^2\right)}\ln E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\left(\gamma C(Y_T)+\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds\right)}\left|Y_t = y\right.\right),$$

$$+\frac{1}{\gamma\left(1-\rho^2\right)}\ln E_{\mathbb{Q}^{mm}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds}\left|Y_t = y\right.\right)$$

and the result follows from Definition 8. □

PROPOSITION 3.1. *The indifference price process is given by*

$$\nu_t(C) = h(Y_t,t) \tag{3.9}$$

where h solves the quasilinear price equation

$$h_t + \mathcal{L}^{me}h - \frac{1}{2}\gamma(1-\rho^2)a^2h_y^2 + c_1(y,t) = 0 \tag{3.10}$$

with

$$h(y,T) = c_2(y).$$

The operator $\mathcal{L}^{me}$ is defined as

$$\mathcal{L}^{me} = \mathcal{L}^{mm} + a^2\frac{u_y}{u}\frac{\partial}{\partial y} \tag{3.11}$$

with u solving (2.16) and $\mathcal{L}^{mm}$ as in (2.8). The price h is the unique $C^{1,2}([0,T]\times\mathcal{R})$ solution of (3.10) and its spatial derivative satisfies $|h_y\ (y,t)| \leq C\left(1+|y|\right)$.

Proof. Equalities (3.3), (3.7) and (3.8) yield

$$\nu_t(C) = h(y,t) = \frac{1}{\gamma\left(1-\rho^2\right)}\ln\frac{u(y,t)}{u^C(y,t)}.$$

Using the equations (2.16) and (3.6), that u and u^C respectively satisfy, yields (3.10). The claimed uniqueness, growth and regularity results follow from Assumptions 1. and 2. and the arguments developed by Pham (Theorems 3.1 and 4.1 (2002)). □

We next derive a probabilistic representation of the indifference price in terms of the minimal relative entropy measure. We first provide an auxiliary result on the density of the latter. The proof is a very mild modification of the one already given by Benth and Kallsen (2003) (see, also, Hobson (2004)).

LEMMA 3.1. *Let u be the solution of (2.16) and assume that $\int_0^T\lambda_s^2ds < \infty$, $\int_0^T a^2(Y,s)\left(\frac{u_y(Y_s,s)}{u(Y_s,s)}\right)^2 ds < \infty$, $\mathbb{P}$ a.s. and, that for some $\alpha > 0$,*

$$E_{\mathbb{P}}\left(e^{\alpha\int_0^T\left(\lambda_s^2 + a^2(Y,s)\left(\frac{u_y(Y_s,s)}{u(Y_s,s)}\right)^2\right)ds}\right) < \infty.$$

Then the density of the minimal relative entropy measure $\mathbb{Q}^{me}$ (2.10) is given by

$$\frac{d\mathbb{Q}^{me}}{d\mathbb{P}} = \exp(-\int_0^T \lambda\left(Y_s,s\right) dW_s^1 + \int_0^T \frac{1}{\sqrt{1-\rho^2}} a\left(Y_s,s\right) \frac{u_y\left(Y_s,s\right)}{u\left(Y_s,s\right)} dW_s^{1,\perp} \tag{3.12}$$

$$-\frac{1}{2}\int_0^T (\lambda^2\left(Y_s,s\right) + \frac{1}{1-\rho^2} a^2\left(Y_s,s\right) \frac{u_y^2\left(Y_s,s\right)}{u^2\left(Y_s,s\right)}) ds) \ .$$

Proof. We first recall a well known fact about the limiting behavior of the indifference price as $\gamma \to 0$ (see, Rouge and El Karoui (2000), Delbaen et al. (2002), Becherer (2003) and Hugonnier et al. (2004)). As the buyer becomes risk neutral, it has been established that the indifference price converges to the expectation of the payoff under the minimal relative entropy measure. In other words, using a generic notation, we have

$$\lim_{\gamma\to 0} \nu_t\left(C;\gamma\right) = \lim_{\gamma\to 0} \mathcal{E}_{\mathbb{Q}^{me}}\left(C\right) = E_{\mathbb{Q}^{me}}\left(C\right).$$

Given the assumptions on the payoff (3.1) and the asset dynamics ((2.1) and (2.2)), it follows that the above limit can be represented as

$$\lim_{\gamma\to 0} \nu_t\left(C;\gamma\right) = E_{\mathbb{Q}^{me}}\left(C\left(Y_s; t\leq s\leq T\right)|Y_t = y\right) = \bar{h}^0\left(y,t\right)$$

for some function $\bar{h}^0$. Using the Feynman-Kac formula, we deduce that $\bar{h}^0$ must solve the linear problem

$$\bar{h}_t^0 + \bar{\mathcal{L}}^{me}\bar{h}^0 + c_1(y,t) = 0$$

with $\bar{h}^0(y,T) = c_2\left(y\right)$ and $\bar{\mathcal{L}}^{me}$ being the generator of the stochastic factor diffusion Y when its dynamics are expressed under the minimal relative entropy measure.

On the other hand, passing to the limit, as $\gamma \to 0$, in (3.10) yields

$$\lim_{\gamma\to 0} h(y,t;\gamma) = h^0(y,t)$$

where h^0 solves the linear problem

$$h_t^0 + (\mathcal{L}^{mm} + a^2\frac{u_y}{u}\frac{\partial}{\partial y})h^0 = h_t^0 + \mathcal{L}^{me}h^0 = c_1(y,t) \tag{3.13}$$

with $h^0(y,T) = c_2(y)$ and $\mathcal{L}^{me}$ introduced in (3.11). The convergence result follows from the robustness properties of viscosity solutions for the equations (3.10) and (3.13) (see Proposition 4.1 in Lions (1984)) and the uniqueness results of the above linear equation. We readily conclude that $\bar{h}^0 = h^0$ for $(y,t) \in \mathcal{R}\times[0,T]$.

The sought density can be then easily identified. The rest of the proof follows from a standard application of Girsanov's theorem, the growth properties of u and the uniqueness of the relative minimal entropy measure (see Benth and Karlsen (Theorem 3.3 (2003)). $\square$

Using the above Lemma, a standard logarithmic transformation in (3.10) (see Musiela and Zariphopoulou (2004a)), and the form of $\mathcal{E}_{\mathbb{Q}^{me}}$, we readily deduce the following representation.

PROPOSITION 3.2. *Let $\mathbb{Q}^{me}$ be the minimal relative entropy measure. Then, the indifference price of the path-dependent claim* $C(Y_s; t \leq s \leq T) = \int_t^T c_1(Y_s, s)ds + c_2(Y_T)$ *admits the representation*

$$\nu_t(C) = h(Y_t, t) \tag{3.14}$$

$$= \mathcal{E}_{\mathbb{Q}^{me}}(C(Y_s : t \leq s \leq T) \, | Y_t = y)$$

$$= -\frac{1}{\gamma(1-\rho^2)} \ln E_{\mathbb{Q}^{me}} \left(e^{-\gamma(1-\rho^2)C(Y_s : t \leq s \leq T)} \, | Y_t = y \right).$$

We note that closed form solutions for indifference prices can be obtained only when the claim's payoff depends exclusively on the stochastic factor. For more general cases, such solutions cannot be obtained and alternative representations need to be sought. Prices have been characterized as solutions to BSDEs (see, for example Rouge and El Karoui (2002), Bielecki et al. (2004)) or as outcomes of iterative valuation algorithms (Becherer (2003), Musiela and Zariphopoulou (2004b and (2005))).

4. Optimal portfolios under CARA preferences; characterization and sensitivity analysis. In this section we analyze the optimal investment strategy of the CARA agent. As in the complete market case (cf. Proposition 2.5), his/her optimal portfolio has two components, the *myopic* and the *non-myopic one.* The myopic investment is the amount the agent would invest in the risky asset if its Sharpe ratio were constant, locally in time, while the non-myopic, or excess risky demand, takes into account the evolution of the investment opportunity set. We concentrate our attention to the excess risky demand and we establish a structural connection with the risk monitoring strategy of an emerging claim. This characterization provides the analogue of (2.28) in an incomplete market framework. The supporting claim is written on the stochastic factor Y, its payoff depends on the stock's Sharpe ratio and the risk aversion coefficient, it is path-dependent and matures at T. Its characteristics are congruous to the ones of its complete market counterpart but the claim is now priced by *indifference.*

We caution the reader that in the indifference valuation context the notion of hedging is not apparent and, therefore, the concept of a risk replicating strategy is not yet well defined. These issues are discussed and addressed in detail below.

i) The supporting claim and its indifference price

We start our analysis by first introducing the supporting claim and constructing its indifference price. We then establish a functional relation between the spatial derivative of its price and the non-myopic portfolio of the CARA agent.

PROPOSITION 4.1. *Consider a path-dependent claim of payoff*

$$\Lambda(Y_s : t \leq s \leq T) = \int_t^T -\frac{1}{2}\frac{\lambda^2\left(Y_s, s\right)}{\gamma} ds \tag{4.1}$$

where S, Y and λ are given by (2.1), (2.2) and (2.6). The claim is introduced at time t and expires at T. Its indifference price $\nu_t\left(\Lambda\right)$ satisfies

$$\nu_t(\Lambda) = h(Y_t, t) \tag{4.2}$$

with $h : \mathcal{R}\times[0, T] \to \mathcal{R}$ given by

$$h(y, t) = \mathcal{E}_{\mathbb{Q}^{me}}\left(\int_t^T -\frac{1}{2}\frac{\lambda^2\left(Y_s, s\right)}{\gamma} ds \left|Y_t = y\right.\right) \tag{4.3}$$

$$= -\mathcal{E}_{\mathbb{Q}^{mm}}\left(\int_t^T \frac{1}{2}\frac{\lambda^2\left(Y_s, s\right)}{\gamma} ds \left|Y_t = y\right.\right). \tag{4.4}$$

ii) The function $h(y, t)$ is the unique $C^{1,2}([0, T] \times \mathcal{R})$ solution of the quasilinear price equation

$$h_t + \frac{1}{2}a^2 h_{yy} + \left(b - \rho\lambda a\right) h_y + \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 = \frac{1}{2\gamma}\lambda^2(y, t) \tag{4.5}$$

with

$$h(y, T) = 0.$$

Moreover, the spatial derivative of h satisfies $|h_y(y, t)| \leq C(1 + |y|)$.

Proof. Using the specific choice of the claim, and the price representations (3.14) and (3.5), we deduce (4.3) and (4.4).

To show part (ii) we first substitute $c_1(y, t) = -\frac{1}{2\gamma}\lambda^2(y, t)$ and $c_2(y) = 0$ in (3.10) to obtain

$$h_t + \frac{1}{2}a^2 h_{yy} + \left(b - \rho\lambda a + a^2\frac{u_y}{u}\right) h_y - \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 = \frac{1}{2\gamma}\lambda^2(y, t) \tag{4.6}$$

and $h(y,T)=0$. We next observe that $u=e^{\gamma(1-\rho^2)h}$. This follows from direct calculations in (2.16) and (3.10). Therefore,

$$\frac{u_y}{u}=\gamma\left(1-\rho^2\right)h_y, \tag{4.7}$$

which combined with the above equation yields (4.5). □

The next result relates the *optimal investment* of the CARA agent with the *spatial derivative* of the indifference price of Λ. Hereafter, this spatial derivative will be referred to as the *indifference hedge.*

THEOREM 4.1. *Let π_s^* be the optimal investment strategy of the CARA agent whose value function is given in (2.5). Then,*

$$\pi_s^*=\pi_s^m+H_s \tag{4.8}$$

where

$$\pi_s^m=\pi^m(Y_s,s)\quad \textit{and}\quad H_s=H(Y_s,s), \tag{4.9}$$

with Y solving (2.2),

$$\pi^m(y,t)=\frac{1}{\gamma}\frac{\mu(y,t)}{\sigma^2(y,t)} \tag{4.10}$$

and

$$H(y,t)=\rho\frac{a(y,t)}{\sigma(y,t)}h_y(y,t). \tag{4.11}$$

The function h is the buyer's indifference price of the path-dependent claim $\Lambda(Y_s;t\le s\le T)$ introduced in (4.1).

Proof. The first order conditions in the HJB equation (2.9) together with the concavity of the value function V with respect to the variable x, yield that the maximum of the involved quadratic occurs at

$$\pi^*(x,y,t)=-\frac{\mu(y,t)}{\sigma^2(y,t)}\frac{V_x(x,y,t)}{V_{xx}(x,y,t)}-\rho\frac{a(y,t)}{\sigma(y,t)}\frac{V_{xy}(x,y,t)}{V_{xx}(x,y,t)}.$$

Using the closed form solution $V(x,y,t)=-e^{-\gamma x}u(y,t)^{1/(1-\rho^2)}$, the above expression further simplifies to

$$\pi^*(x,y,t)=\frac{\mu(y,t)}{\gamma\sigma^2(y,t)}+\rho\frac{a(y,t)}{\sigma(y,t)}\frac{1}{\gamma(1-\rho^2)}\frac{u_y(y,t)}{u(y,t)} \tag{4.12}$$

where u solves the linear problem (2.16). Classical verification results together with the relevant regularity and growth properties yield the optimality of the proposed policy (see Pham (2002) and Zariphopoulou (2003)). The rest of the proof follows from (4.7). □

ii) Indifference risk monitoring strategies and non-myopic optimal investments

We recall that the concept of indifference price is based on the optimality of investment opportunities with and without the claim. It is then natural to start our analysis with the behavior of these optimal portfolios. They are denoted by $\pi_s^{\Lambda,*}$ and $\pi_s^{0,*}$ and represent the optimal amounts the buyer holds with and without the claim.

We recall that if the claim is not bought, the buyer starts at t with wealth x and follows the policy $\pi_s^{0,*}$. If he/she buys the claim at time t, the initial wealth and the optimal policy are, respectively, $x - \nu_t(\Lambda)$ and $\pi_s^{\Lambda,*}$.

The optimal wealth trajectories, $X_s^{\Lambda,*}$ and $X_s^{0,*}$ solve

$$dX_s^{\Lambda,*} = \mu(Y_s, s)\pi_s^{\Lambda,*}ds + \sigma(Y_s, s)\pi_s^{\Lambda,*}dW_s^1, \tag{4.13}$$

and

$$dX_s^{0,*} = \mu(Y_s, s)\pi_s^{0,*}ds + \sigma(Y_s, s)\pi_s^{0,*}dW_s^1 \tag{4.14}$$

with

$$X_t^{\Lambda,*} = x - \nu_t(\Lambda) \quad \text{and} \quad X_t^{0,*} = x. \tag{4.15}$$

Proposition 4.2. *Let $\pi_s^{\Lambda,*}$ and $\pi_s^{0,*}$ be, respectively, the optimal amounts invested in the stock account that the buyer holds with and without the claim Λ. Let h be its indifference price given in (4.2). Then, for $t \leq s \leq T$,*

$$\pi_s^{\Lambda,*} - \pi_s^{0,*} = -\rho\frac{a(Y_s, s)}{\sigma(Y_s, s)}h_y(Y_s, s). \tag{4.16}$$

Proof. We recall that the buyer's value function is given by

$$V^\Lambda(x, y, t) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_T + \int_t^T\left(-\frac{1}{2}\frac{\lambda^2 Y_s, s)}{\gamma}\right)ds\right)}|Y_t = y\right).$$

Applying standard verification results we obtain

$$\pi_s^{\Lambda,*} = -\frac{\mu(Y_s, s)}{\sigma^2(Y_s, s)}\frac{V_x^\Lambda(X_s^{\Lambda,*}, Y_s, s)}{V_{xx}^\Lambda(X_s^{\Lambda,*}, Y_s, s)} - \rho\frac{a(Y_s, s)}{\sigma(Y_s, s)}\frac{V_{xy}^\Lambda(X_s^{\Lambda,*}, Y_s, s)}{V_{xx}^\Lambda(X_s^{\Lambda,*}, Y_s, s)}$$

with $X_s^{\Lambda,*}$ given in (4.13). On the other hand, the definition of the indifference price implies

$$V^\Lambda(x, y, t) = V(x + h(y, t), y, t)$$

$$= -e^{-\gamma(x+h(y,t))}u\,(y, t)^{1/\left(1-\rho^2\right)}$$

with u solving (2.16). Combining the above yields

$$\pi_s^{\Lambda,*} = \frac{1}{\gamma}\frac{\mu(Y_s,s)}{\sigma^2(Y_s,s)} - \rho\frac{a(Y_s,s)}{\sigma(Y_s,s)}\left(h_y(Y_s,s) - \frac{1}{\gamma(1-\rho^2)}\frac{u_y(Y_s,s)}{u(Y_s,s)}\right). \quad (4.17)$$

Clearly, $\pi_s^{0,*} = \pi_s^*$, with the latter given in (4.8) (see, also, (4.10) and (4.11)), and rewritten below,

$$\pi_s^{0,*} = \frac{1}{\gamma}\frac{\mu(Y_s,s)}{\sigma^2(Y_s,s)} + \rho\frac{a(Y_s,s)}{\sigma(Y_s,s)}\frac{1}{\gamma(1-\rho^2)}\frac{u_y(Y_s,s)}{u(Y_s,s)}.$$

Subtracting the above expressions for $\pi_s^{\Lambda,*}$ and $\pi_s^{0,*}$ yields (4.16). □

We next introduce two important quantities that will help us develop the appropriate notion of the *indifference risk monitoring* strategy. These are the *residual optimal wealth* and the *residual risk*, denoted respectively by L_s and R_s. In the context of indifference valuation and risk quantification, these processes were introduced by Musiela and Zariphopoulou (2001 and 2004a).

Definition 4.1. *The residual optimal wealth process L_s is defined, for $t \le s \le T$, by*

$$L_s = X_s^{0,*} - X_s^{\Lambda,*} \quad (4.18)$$

where $X_s^{\Lambda,*}$ and $X_s^{0,*}$ are the optimal wealth processes (4.13) and (4.14).

Definition 4.2. *The residual risk process R_s is defined, for $t \le s \le T$, by*

$$R_s = \nu_s(\Lambda) - L_s \quad (4.19)$$

where the processes $\nu_s(\Lambda)$ and L_s are, respectively, the indifference price and the optimal residual wealth.

In complete markets, the arbitrage free price and the indifference price are identical. Direct arguments imply that the buyer's residual optimal wealth coincides with the classical hedging portfolio. The residual risk is naturally eliminated due to perfect replication. In incomplete markets, however, this is not the case. The claim's payoff cannot be entirely reproduced by trading the tradable assets and, thus, it differs from the residual optimal wealth. The claim is only partially replicated and the residual risk is different than zero.

The following results is a direct consequence of Definition 4.1 and Propostion 4.2.

Lemma 4.1. *The residual optimal wealth process L_s satisfies, for $t \le$*

$s \leq T$,

$$dL_s = \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s) \left(\mu(Y_s, s) ds + \sigma(Y_s, s) dW_s^1 \right) \tag{4.20}$$

$$= \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s) \frac{dS_s}{S_s},$$

with

$$L_t = h(Y_t, t) = \nu_t(\Lambda), \tag{4.21}$$

for $t \leq s \leq T$, *and* h *given in (4.2).*

We remind the reader that L_s is the residual optimal wealth of the buyer of the claim Λ while H_s is the excess risky demand that the investor follows in the optimal portfolio choice problem (2.5). The following payoff decomposition result establishes the connection between these two processes.

THEOREM 4.2. *Let* H_s *and* Λ_s *be, respectively, the non-myopic portfolio process of the CARA agent in the expected utility maximization problem (2.5) and the residual optimal wealth of the buyer of* Λ.

i) The residual optimal wealth is given, for $s \leq t \leq T$, *by*

$$L_s = \nu_t(\Lambda) + \int_t^s H_u \frac{dS_u}{S_u}. \tag{4.22}$$

ii) The path-dependent claim Λ *admits, under the historical measure* $\mathbb{P}$, *the payoff decomposition*

$$\Lambda(Y_s; t \leq s \leq T) = \int_t^T -\frac{1}{2} \frac{\lambda^2(Y_s, s)}{\gamma} ds = \nu_t(\Lambda) + \int_t^T H_s \frac{dS_s}{S_s}$$

$$- \int_t^T \frac{1}{2} \gamma (1 - \rho^2) a^2(Y_s, s) h_y^2(Y_s, s) ds + \int_t^T \sqrt{1 - \rho^2} a(Y_s, s) h_y(Y_s, s) dW_s^{1,\perp} \tag{4.23}$$

where the indifference price h *solves (4.5).*

Proof. We first observe that Theorem 4.1, together Lemma 4.1, yield

$$dL_s = H_s \frac{dS_s}{S_s}$$

and part (i) follows.

To show (ii), we use Definition 4.2 and the price equation (4.5) to obtain

$$dR_s = \frac{1}{2} \left(-\gamma (1 - \rho^2) a^2(Y_s, s) h_y^2(Y_s, s) + \frac{\lambda^2(Y_s, s)}{\gamma} \right) ds$$

$$-\rho a(Y_s,s)h_y(Y_s,s)dW_s^1+a(Y_s,s)h_y(Y_s,s)dW_s,$$

or, equivalently,

$$dR_s=\frac{1}{2}\left(-\gamma(1-\rho^2)a^2(Y_s,s)h_y^2(Y_s,s)+\frac{\lambda^2(Y_s,s)}{\gamma}\right)ds$$

$$+\sqrt{1-\rho^2}a(Y_s,s)h_y(Y_s,s)dW_s^{1,\perp}.$$

Moreover,

$$R_t=\nu_t\left(C\right)-L_t=0 \quad \text{and} \quad R_T=h(Y_T,T)-L_T=-L_T.$$

Therefore,

$$R_T=\int_t^T\left(-\frac{1}{2}\gamma(1-\rho^2)a^2(Y_s,s)h_y^2(Y_s,s)+\frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}\right)ds$$

$$+\int_t^T\sqrt{1-\rho^2}a(Y_s,s)h_y(Y_s,s)dW_s^{1,\perp}$$

which, combined with the above, yields

$$\Lambda(Y;t\leq s\leq T)=\int_t^T-\frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}ds$$

$$=L_T-\int_t^T\frac{1}{2}\gamma(1-\rho^2)a^2(Y_s,s)h_y^2(Y_s,s)ds+\int_t^T\sqrt{1-\rho^2}a(Y_s,s)h_y(Y_s,s)dW_s^{1,\perp}.$$

Using (4.22) we conclude. ▯

We next turn our attention to properties of the price process, the residual optimal wealth and the residual risk in terms of the pricing measure.

PROPOSITION 4.3. *Let $\mathbb{Q}^{me}$ be the minimal relative entropy measure and $\mathcal{F}_s$ the enlarged filtration of the (S,Y) model. With respect to this measure and filtration,*

i) the indifference price process $\nu_s\left(\Lambda\right)$ and the residual risk R_s are submartingales.

ii) the residual optimal wealth L_s is a martingale; moreover, it is a martingale under all $Q\in\mathcal{Q}_e$.

Proof. We recall that the indifference price is given by $\nu_s\left(\Lambda\right)=h(Y_s,s)$ with h solving (3.10). We next consider the dynamics of the state process S and Y under $\mathbb{Q}^{me}$, namely,

$$dS_s=\sigma(Y_s,s)d\tilde{W}_s^{1,me}$$

and

$$dY_s = \left(b(Y_s, s) - \rho\lambda(Y_s, s)a(Y_s, s) + a^2(Y_s, s)\frac{u_y(Y_s, s)}{u(Y_s, s)} \right) ds + a(Y_s, s)d\tilde{W}_s^{me},$$

where

$$d\tilde{W}_s^{1,me} = dW_s^1 + \lambda(Y_s, s)ds$$

and

$$d\tilde{W}_s^{me} = dW_s - a(Y_s, s)\frac{u_y(Y_s, s)}{u(Y_s, s)}ds + \rho\lambda(Y_s, s)ds$$

are standard Brownian motions under $\mathbb{Q}^{me}$ having correlation ρ. We then have

$$\tilde{W}_s^{me} = \rho W_s^{1,me} + \sqrt{1-\rho^2}\; W_s^{1,me,\perp}$$

where $W_s^{1,me,\perp}$ is orthogonal to $W_s^{1,me}$ under $\mathbb{Q}^{me}$.

Using the regularity properties of h and Ito's formula, we then deduce

$$dh(Y_s, s) = \frac{1}{2}\left(\gamma(1-\rho^2)a^2(Y_s, s)h_y^2(Y_s, s) + \frac{\lambda^2(Y_s, s)}{\gamma} \right) ds$$

$$+ a(Y_s, s)h_y(Y_s, s)d\tilde{W}_s^{me}.$$

The submartingale property then follows from the positivity of the above drift and the regularity and growth properties of h (cf. Proposition 4.1).

The martingale property of L_s is a direct consequence of (4.22) while the submartingale property of the residual risk follows from its definition and the properties of $\nu_s(\Lambda)$ and L_s. □

Theorem 4.3. *Let Λ be the path-dependent claim*

$$\Lambda = \int_t^T -\frac{1}{2}\frac{\lambda^2(Y_s, s)}{\gamma}ds$$

and h its indifference price function.

i) Its residual risk process R_s admits, under $\mathbb{Q}^{me}$, the Doob decomposition

$$R_s = M_s^R + A_s^R$$

where

$$M_s^R = \int_t^s a(Y_u, u)h_y(Y_u, u)\sqrt{1-\rho^2}\; d\tilde{W}_u^{1,me,\perp}$$

and

$$A_s^R = \int_t^s \frac{1}{2}\left(\gamma(1-\rho^2)a^2(Y_u,u)h_y^2(Y_u,u) + \frac{\lambda^2(Y_u,u)}{\gamma}\right)du.$$

ii) The indifference price process $\nu_s(\Lambda)$ admits, under $\mathbb{Q}^{me}$, the Doob decomposition

$$\nu_s(\Lambda) = M_s + A_s^R$$

where

$$M_s = L_s + M_s^R$$

$$= \nu_t(\Lambda) + \int_t^s H_u \frac{dS_u}{S_u} + \int_t^s a(Y_u,u)h_y(Y_u,u)\sqrt{1-\rho^2}\, d\tilde{W}_u^{1,me,\perp}.$$

The process M_s^R is a $\mathbb{Q}^{me}$-martingale while the process A_s^R is a non-decreasing predictable process.

iii) Sensitivity analysis of non-myopic portfolios

We continue with the sensitivity analysis for the non-myopic feedback portfolio policy,

$$H(y,t) = \rho\frac{a}{\sigma}h_y(y,t). \tag{4.24}$$

Specifically, we are interested in its sign and its behavior in terms of the trading horizon. Some of these issues have been analyzed by various authors but under restrictive model assumptions (see, among others, Kim and Omberg (1996), Liu (1999) and Wachter (2002)).

For the sake of presentation and conciseness of results, it is throughout assumed that the market coefficients μ, b and σ are autonomous functions of the discount factor and that $a = 1$. The case of non constant volatility coefficient a is easily addressed via a well known drift transformation (see, Friedman (1975) and Section 5 in Musiela and Zariphopoulou (2001) for related comments on model specification and indifference prices). The state process Y then solves

$$dY_s = b(Y_s)ds + dW_s. \tag{4.25}$$

We start with two preliminary probabilistic results for the *indifference delta* function h_y.

Lemma 4.2. *Let $\mathbb{Q}^{me}$ and $\mathbb{Q}^{mm}$ be, respectively, the minimal relative entropy and the minimal martingale measure in the incomplete market* *model* $(S, Y\ )$ *where* S *and* Y *solve, respectively, (2.1) and (4.25). Let also* u *be the solution of*

$$u_t + \frac{1}{2}u_{yy} + cu_y = \frac{1}{2}(1-\rho^2)\lambda^2 u \tag{4.26}$$

with $c(y) = b(y) - \rho\lambda(y)$*, and* h *be the indifference price of* Λ *solving (4.5).*

The indifference delta function $\delta = h_y$ *admits the probabilistic representations*

$$\delta(y,t) = -\frac{1}{\gamma}E_{\mathbb{Q}^{me}}\left(\int_t^T e^{c_y(Y_s,s)}\lambda(Y_s)\lambda_y(Y_s)ds\,|Y_t = y\right) \tag{4.27}$$

and

$$\delta(y,t) = -\frac{1}{\gamma}E_{\mathbb{Q}^{mm}}\left(\int_t^T e^{Z_s}\lambda(Y_s)\lambda_y(Y_s)\frac{u(Y_s,s)}{u(y,t)}ds\,|Y_t = y\right) \tag{4.28}$$

where

$$Z_s = \int_t^s \left(c_y(Y_u) - \frac{1}{2}(1-\rho^2)\lambda^2(Y_u)\right)du.$$

Proof. We recall that h solves

$$h_t + \frac{1}{2}h_{yy} + ch_y + \frac{1}{2}\gamma(1-\rho^2)h_y^2 = \frac{1}{2\gamma}\lambda^2(y).$$

Direct differentiation yields that $\delta = h_y$ solves the *viscous Burger's* equation

$$h_{yt} + \mathcal{L}^{mm}h_y + \gamma\left(1-\rho^2\right)h_y h_{yy} + c_y h_y = \frac{1}{\gamma}\lambda\lambda_y \tag{4.29}$$

with $h_y(y,T) = 0$. Representation (4.27) then follows from the Feynman-Kac formula.

To establish (4.28), we first recall (4.7). Differentiating (4.26) we obtain

$$u_{yt} + \mathcal{L}^{mm}u_y + \left(c_y - \frac{1}{2}\left(1-\rho^2\right)\lambda^2\right)u_y = (1-\rho^2)\lambda\lambda_y u$$

with $u_y(y,T) = 0$. The Feynman-Kac formula then yields

$$u_y(y,t) = E_{\mathbb{Q}^{mm}}\left(-\int_t^T e^{-Z_s}(1-\rho^2)\lambda(Y_s,s)\lambda_y(Y_s,s)u(Y_s,s)ds\,|Y_t = y\right),$$

and the result follows. ◻

LEMMA 4.3. *Let π^m and H be respectively the optimal myopic and non-myopic investment feedback law functions (4.10) and (4.11). Then H solves*

$$H_t + \frac{1}{2}H_{yy} + k(y,t)H_y + K(y,t)H = \rho\pi^m\lambda_y \tag{4.30}$$

with $H(y,T) = 0$,

$$k(y,t) = \frac{\sigma_y(y)}{\sigma(y)} + \frac{u_y(y,t)}{u(y,t)} + c(y),$$

$$K(y,t) = c(y)\frac{\sigma_y(y)}{\sigma(y)} + c_y(y) + \frac{\sigma_{yy}(y)}{2\sigma(y)} + \frac{\sigma_y(y)}{\sigma(y)}\frac{u_y(y,t)}{u(y,t)}$$

and u solving (4.26).

Proof. The proof follows directly from the feedback expressions (4.10) and (4.11), for π^m and H, and from (4.29). □

The result below is a direct consequence of the above Lemma and the positivity of u.

PROPOSITION 4.4. *i) If the Sharpe ratio λ is positive and non-decreasing in y, the excess risky demand H is positive (negative) if the stock and the stochastic factor are negatively (positively) correlated.*

ii) If the Sharpe ratio λ is positive and non-increasing in y, the excess risky demand H is positive (negative) if the stock and the stochastic factor are positively (negatively) correlated.

The next result examines the behavior of the excess risky demand in terms of the investment horizon.

PROPOSITION 4.5. *i) If the Sharpe ratio λ is positive and non-decreasing in y, the excess risky demand H is non-increasing (non-decreasing) with respect to the investment horizon $T-t$, if the stock and the stochastic factor are negatively (positively) correlated.*

ii) If the Sharpe ratio λ is positive and non-increasing in y, the excess risky demand H is non-decreasing (non-increasing) with respect to the investment horizon $T-t$, if the stock and the stochastic factor are positively (negatively) correlated.

Proof. Let

$$f(y,t) = \frac{u_y(y,t)}{u(y,t)} \quad \text{and} \quad g(y,t) = f_t(y,t).$$

Direct calculations in (4.26) yield that g solves

$$g_t + \frac{1}{2}g_{yy} + (c+f)g_y + g\,(c_y + f_y) = 0, \quad (y,t) \in \mathcal{R}\times[0,T]$$

with

$$g(y,T) = \left(1-\rho^2\right)\lambda(y)\lambda_y(y).$$

The comparison principle then yields that if λ is positive and non-decreasing (non-increasing) function of the stochastic factor level, then $g \geq 0$ ($g \leq 0$). The result then follows from (4.24). □

The behavior of the excess risky demand in terms of the *level* of the stochastic factor and the *correlation* may be obtained from (4.30). For example, differentiating with respect to y yields that the function $F = H_y$ solves the linear problem

$$F_t + \frac{1}{2}F_{yy} + k(y,t)F_y + m(y,t)F + M(y,t) = 0$$

with $F(y,T) = 0$, where

$$m(y,t) = k_y(y,t) + K(y,t)$$

and

$$M(y,t) = K_y(y,t)H - \frac{\rho}{\gamma}\left(\frac{\lambda\lambda_y}{\sigma(y)}\right)_y .$$

The sign of H_y will then be the same as the one of M. However, it is difficult to obtain general results given the complexity of the above coefficients. The same is also true for the case of correlation due to the way correlation affects the involved terms, h_y and h_{yy} among others.

5. Equivalent classes under incompleteness. In this section we explore the similarities between two classes of expected utility models. The first class is the one we studied in the previous section. The goal was to maximize expected utility of terminal wealth in a market environment in which the stock dynamics are affected by a correlated stochastic factor and the risk preferences are exponential. The second class, introduced below, contains models with lognormal stock dynamics but with path-dependent preferences. The utility is of a (multiplicative) separable form. Its first component depends solely on the terminal wealth while the second on a compounding term involving an integral functional of a process correlated with the lognormal stock. In both models, a second asset is available for trading, a deterministic bond offering zero interest rate. The two models are equivalent in the sense that their value functions coincide.

Class P_1 : The model is the one introduced in Section 2. For convenience we recall that the stock dynamics are given by,

$$dS_s = \mu(Y_s,s)S_s ds + \sigma(Y_s,s)S_s dW_s^1, \tag{5.1}$$

where the stochastic factor solves

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s. \tag{5.2}$$

The processes W^1 and W are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and they have correlation $\rho \in (-1, 1)$. The state wealth process is given by

$$dX_s = \mu(Y_s, s)\pi_s ds + \sigma(Y_s, s)\pi_s dW_s^1. \tag{5.3}$$

The utility depends only on terminal wealth and is of exponential type, namely,

$$U(X_T) = -e^{-\gamma X_T} \quad \text{with } \gamma > 0.$$

The value function is

$$V(x, y, t) = \sup_{\mathcal{A}} E(U(X_T)\,|X_t = x, Y_t = y) \tag{5.4}$$

for $(x, y, t) \in \mathcal{D} = \mathcal{R} \times \mathcal{R} \times [0, T]$.

Class $\mathcal{P}_2$: The stock dynamics are lognormal

$$d\tilde{S}_s = \tilde{\mu}\tilde{S}_s ds + \ \tilde{\sigma}\tilde{S}_s d\tilde{W}_s^1 \tag{5.5}$$

with $\tilde{\mu}$ and $\tilde{\sigma}$ being positive constants. The state wealth satisfies

$$d\tilde{X}_s = \tilde{\mu}\tilde{\pi}_s ds + \tilde{\sigma}\tilde{\pi}_s d\tilde{W}_s^1 \tag{5.6}$$

where $\tilde{\pi}_s$ stands for the amount invested in the stock. The agent's preferences are of the form

$$\tilde{U}((\tilde{X}_T, \tilde{Y}_s); t \leq s \leq T) = \Phi(\tilde{X}_T)\exp(\int_t^T -\frac{1}{2}L^2(\tilde{Y}_s, s)ds), \tag{5.7}$$

with $\tilde{Y}$ solving

$$d\tilde{Y}_s = \tilde{b}(\tilde{Y}_s, s)ds + \tilde{a}(\tilde{Y}_s, s)d\tilde{W}_s. \tag{5.8}$$

The processes $\tilde{W}^1$ and $\tilde{W}$ are standard Brownian motions defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and they have correlation $\rho \in (-1, 1)$ as in model $\mathcal{P}_1$.

The value function $\tilde{V}$ is

$$\tilde{V}(x, y, t) = \sup_{\mathcal{A}} E(\tilde{U}(\tilde{X}_T, \tilde{Y}_s); t \leq s \leq T)\,\Big|\tilde{X}_t = x, \tilde{Y}_t = y) \tag{5.9}$$

for $(x, y, t) \in \mathcal{D}$. We denote (with a slight abuse of notation) by $\mathcal{A}$ the set of admissible policies, defined as $\mathcal{A}$ =$\{\tilde{\pi} : \tilde{\pi}_s$ is $\tilde{\mathcal{F}}_s-$ measurable and

$E\int_t^T \tilde{\pi}_s^2 ds < \infty\}$ where $\tilde{\mathcal{F}}_s = \{\sigma\left(\tilde{W}_u^1, \tilde{W}_u) : t \le u \le s\right)\}$. The constant Sharpe ratio is

$$\tilde{\lambda} = \frac{\tilde{\mu}}{\tilde{\sigma}}.$$

We next consider the minimal relative entropy measure, $\tilde{\mathbb{Q}}^{me}$, associated with this model. It easily follows

$$\tilde{\mathbb{Q}}^{me}(A) = E_{\tilde{\mathbb{P}}}\left(e^{-\tilde{\lambda}W_T^1 - \frac{1}{2}\tilde{\lambda}^2 T}\mathbf{1}_A\right), \quad A \in \tilde{\mathcal{F}}_T \tag{5.10}$$

for $A \in \tilde{\mathcal{F}}_T$.

THEOREM 5.1. *Assume that, for $(x, y, t) \in \mathcal{R} \times \mathcal{R} \times [0, T]$, the market coefficients and the utility functionals of models $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy*

$$a(y,t) = \tilde{a}(y,t) \tag{5.11}$$

$$b(y,t) - \rho\lambda(y,t)a(y,t) = \tilde{b}(y,t) - \rho\tilde{\lambda}a(y,t) \tag{5.12}$$

$$L^2(y,t) = \lambda^2(y,t) - \tilde{\lambda}^2(T-t) \tag{5.13}$$

and

$$\Phi(x) = U(x). \tag{5.14}$$

Let $\mathbb{Q}^{me}$ and $\tilde{\mathbb{Q}}^{me}$ be the minimal relative entropy measures (2.10) and (5.10).

Then the value functions V and $\tilde{V}$ ((5.4) and (5.9)) coincide,

$$V(x,y,t) = \tilde{V}(x,y,t), \qquad (x,y,t) \in \mathcal{D};$$

they are given by

$$V(x,y,t) = -e^{-\gamma x} E_{\mathbb{Q}^{me}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)ds} | Y_t = y)\right)^{1/(1-\rho^2)}$$

and

$$\tilde{V}(x,y,t) = -e^{-\gamma x} E_{\tilde{\mathbb{Q}}^{me}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(\tilde{Y}_s,s)ds} \middle| \tilde{Y}_t = y)\right)^{1/(1-\rho^2)}$$

with the factors Y and $\tilde{Y}$ solving (5.2) and (5.8).

Proof. The result follows from the scaling properties of the value functions, the uniqueness of solutions to their associated HJB equations and the appropriate choice of market coefficients. Indeed, let us recall that the first value function, V, is given by $V(x,y,t) = -e^{-\gamma x} u(y,t)^{1/(1-\rho^2)}$, where u solves

$$u_t + \frac{1}{2} a^2(y,t) u_{yy} + \left((b(y,t) - \rho\lambda(y,t) a(y,t) \right) u_y = \frac{1}{2}(1-\rho^2)\lambda^2(y,t) u \quad (5.15)$$

with $u(y,T) = 1$ (cf. ((2.16))).

Similar arguments yield, after direct albeit tedious calculations, that the second value function can be represented as

$$\tilde{V}(x,y,t) = -e^{-\gamma x} \tilde{u}(y,t)^{1/(1-\rho^2)}$$

where $\tilde{u}$ solves

$$\tilde{u}_t + \frac{1}{2} a^2(y,t) \tilde{u}_{yy} + \left(\tilde{b}(y,t) - \rho\tilde{\lambda} a(y,t) \right) \tilde{u}_y = \frac{1}{2}(1-\rho^2) \left(L^2(y,t) + \tilde{\lambda}^2 (T-t) \right) \tilde{u} \quad (5.16)$$

with $\tilde{u}(x,y,t) = 1$.

Using the structural assumptions on the market coefficients (see (5.11), (5.12), (5.13) and (5.13)) and the uniqueness of solutions to the above two linear pdes, we easily conclude. □

We next apply the previous results to the derivation of indifference prices of claims written on *non-traded* assets under the assumption of lognormal stock prices. The claim is introduced at time t and matures at T, yielding payoff

$$\tilde{C}(\tilde{Y}_s; t \le s \le T) = \int_t^T \tilde{c}_1 \left(\tilde{Y}_s, s \right) ds + \tilde{c}_2 \left(\tilde{Y}_T \right). \quad (5.17)$$

This class of derivatives is different than the one considered in Section 3 in that the levels of the non-traded asset do not directly affect the dynamics of the traded one. Indifference prices in such settings have been so far analyzed for European claims (see, Henderson (2002), Musiela and Zariphopoulou (2004)), for early exercise contracts (see, Musiela and Zariphopoulou (2004c), Oberman and Zariphopoulou (2003), Sokolova (2004)) and for perpetual options (Henderson (2004b)).

The results herein can be used to produce indifference prices of claims as in (5.17). Such claims appear in the valuation of stochastic labor income stream (see Henderson (2004), for a special case). The emerging valuation formula (see (5.18) below) is consistent with (3.14) even though market incompleteness is being introduced very differently. In both market settings, the indifference price is given in terms of a nonlinear valuation functional of the payoff, and the pricing measure is the minimal relative entropy one. Note that when the stock dynamics are lognormal, the minimal martingale and the minimal relative entropy measure are identical, namely,

$$\tilde{\mathbb{Q}}^{me} = \tilde{\mathbb{Q}}^{mm} = \tilde{\mathbb{Q}}$$

and a single nonlinear indifference valuation formula emerges.

PROPOSITION 5.1. *Consider a model in which a traded and a non-traded asset have prices $\tilde{S}$ and $\tilde{Y}$, given by (5.5) and (5.8). Let $\mathcal{E}_Q$ be the nonlinear pricing functional of the form (3.4). The buyer's indifference price of the claim $\tilde{C}(\tilde{Y}_s; t \le s \le T)$ is given by*

$$\nu_t(\tilde{C}(\tilde{Y}_s; t \le s \le T)) = \mathcal{E}_{\tilde{\mathbb{Q}}}(\tilde{C}(\tilde{Y}_s; t \le s \le T)) \tag{5.18}$$

$$= -\frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{\mathbb{Q}}} \left(e^{-\gamma(1-\rho^2)\left(\int_t^T \tilde{c}_1(Y_s,s)ds + \tilde{c}_2(Y_T)\right)} \Big| \tilde{Y}_t = y \right).$$

The measure $\tilde{\mathbb{Q}}$ is the martingale measure that has the minimal, relative to $\tilde{\mathbb{P}}$, entropy.

Proof. We first recall that in the absence of the claim, the value function $\tilde{V}$ is the solution to the classical Merton's model and is given by

$$\tilde{V}(x,t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma \tilde{X}_T} \Big| \tilde{X}_t = x \right) = -e^{-\gamma x - \frac{1}{2}\tilde{\lambda}^2 (T-t)} \tag{5.19}$$

(see Merton (1990)). If the claim is bought, the buyer's value function is

$$\tilde{V}^{\tilde{C}}(x,y,t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma\left(\tilde{X}_T + \tilde{C}(\tilde{Y}_s; t \le s \le T)\right)} \Big| \tilde{X}_t = x, \tilde{Y}_t = y \right)$$

where $\tilde{X}_s$ solves (5.6). We can now use the results of Theorem 5.1 for the choice of preferences

$$\tilde{U}\left(\tilde{X}_T, \tilde{Y}_s; t \le s \le T\right) = e^{-\gamma \tilde{X}_T} e^{-\gamma \tilde{C}(\tilde{Y}_s; t \le s \le T)}.$$

Direct calculations yield

$$\tilde{V}^{\tilde{C}}(x,y,t) = -e^{-\gamma x} \tilde{u}^{\tilde{C}}(y,t)^{1/\left(1-\rho^2\right)}$$

where

$$\tilde{u}^{\tilde{C}}(y,t) = E_{\tilde{\mathbb{Q}}} \left(e^{-\gamma\left(1-\rho^2\right)\left(\tilde{C}(\tilde{Y}_s; t \le s \le T) + \frac{1}{2}\frac{\tilde{\lambda}}{\gamma}^2 (T-t)\right)} | Y_t = y \right).$$

The result then follows from Definition 3.1 and (5.19). □

6. Conclusions. We analyzed the optimal investment strategies of a CARA agent who maximizes expected utility of terminal wealth. The optimal portfolio consists of two components, the myopic and the non-myopic one. A structural characterization of the latter was presented for the latter, yielding the excess risky demand as the risk monitoring strategy of an emerging supporting claim. The claim is path-dependent, written on the Sharpe ratio and the risk tolerance, and is priced by indifference.

The notion of indifference risk monitoring strategies was developed via the associated optimal policies, the residual optimal wealth and the residual risk of the buyer.

An important extension is to allow for intermediate consumption. This would naturally alter the nature of the supporting claim. Due to the consumption stream, a dividend type payoff component is expected to emerge in addition to the path-dependent part that contains the market price of risk. A second direction is to allow for risk preferences different than exponential. The interesting question would then be what is the composition of the supporting claim and what would be the appropriate valuation method. At first, one might think that for other preferences, the valuation should be also done by indifference but with the involved pricing conditions appropriately modified to reflect the preference choice. However, this might not be the case. If, for example, the investor's utility if of CRRA type, preliminary results show that the emerging claim is priced by indifference, but as if the buyer still had CARA preferences in an embedded model (see Stoikov (2004)).

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