Computing Time-Dependent Bid-Prices in Network Revenue Management Problems

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Abstract

We present a new method for computing bid-prices in network revenue management problems. The novel aspect of our method is that it naturally provides bid-prices that depend on how much time is left until the departure. Our method provides an upper bound on the optimal objective value of the problem and this upper bound is tighter than the one obtained by the so-called deterministic linear program. Similar to the bid-prices obtained by the deterministic linear program, the bid-prices obtained by our method are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. When applied on network revenue management problems with cancellations, our method formally shows how the deterministic linear program should be modified to incorporate cancellations. Computational experiments indicate that our method can improve on other solution methods that are used to solve network revenue management problems in practice.

Bid-prices form a powerful tool for constructing good policies for network revenue management problems. The fundamental idea is to associate a bid-price with each flight leg, capturing the opportunity cost of a unit of capacity. An itinerary request is accepted only when the revenue from the requested itinerary exceeds the sum of the bid-prices of the flight legs in the requested itinerary; see Williamson (1992), Talluri and van Ryzin (1998) and Talluri and van Ryzin (2004).

Traditionally, bid-prices are computed by solving a deterministic linear program. This deterministic linear program has one constraint for each flight leg and the right side of these constraints are the remaining leg capacities. Therefore, the optimal values of the dual variables associated with these constraints are used as bid-prices. A shortcoming of the deterministic linear program, however, is that it uses only the expected numbers of the itinerary requests that are to arrive until the departure and does not incorporate the probability distributions or temporal dynamics of the arrivals of the itinerary requests. In practice, as the itinerary requests arrive and the remaining leg capacities are adjusted, the deterministic linear program is periodically resolved to artificially incorporate the temporal dynamics of the arrivals of the itinerary requests.

In this paper, we present a new method for computing bid-prices. The deterministic linear program described above is viewed as a straightforward deterministic approximation, whereas our method directly works with the dynamic programming formulation of the network revenue management problem. The idea behind our method is to relax the capacity availability constraints in the dynamic programming formulation of the network revenue management problem by associating Lagrange multipliers with them. In this case, the optimality equation decomposes by the time periods and we obtain a simple expression for the value function. A good set of Lagrange multipliers can easily be obtained by minimizing a convex function. Since the Lagrange multipliers depend on how much time is left until the departure, the bid-prices that we obtain in this fashion also depend on how much time is left until the departure and our method partially incorporates the temporal dynamics of the arrivals of the itinerary requests.

When compared with the other solution methods in the network revenue management literature, the approach that we follow in this paper provides several advantages. First, computational experiments indicate that the bid-prices obtained by our method perform noticeably better than the ones obtained by the deterministic linear program. Second, our method provides an upper bound on the optimal objective value of the network revenue management problem. Although Bertsimas and Popescu (2003) show that the deterministic linear program also provides such an upper bound, the upper bound obtained by our method is provably tighter than the one obtained by the deterministic linear program. Third, the bid-prices obtained by our method are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. This property provides some theoretical basis for using our method and it is also known to hold for the bid-prices obtained by the deterministic linear program; see Talluri and van Ryzin (1998). Fourth, when our method is applied on network revenue management problems with cancellations, it formally shows how the deterministic linear program should be modified to incorporate cancellations. This strengthens the links between the dynamic programming formulation of the network revenue management problem and the deterministic linear program. The form of the deterministic linear program is well-known under the assumption that there are no cancellations, but it is not clear how to modify the deterministic linear program to incorporate cancellations and our method fills this gap.

Network revenue management has been an active area of research for the past two decades. Simpson (1989) and Williamson (1992) were the first to use the deterministic linear program to compute bid-prices. Talluri and van Ryzin (1998) show the asymptotic optimality result mentioned above for the bid-prices obtained by the deterministic linear program. They use a relaxation strategy that resembles our method, but their Lagrange multipliers do not depend on how much time is left until the departure. As a result, they do not naturally incorporate the temporal dynamics of the arrivals of the itinerary requests. Talluri and van Ryzin (1999) propose a randomized version of the deterministic linear program that uses actual samples of the numbers of the itinerary requests that are to arrive until the departure. Bertsimas and Popescu (2003) compute bid-prices by using the change in the optimal objective value of the deterministic linear program induced by a change in the right sides of certain constraints. In this way, they try to capture the total opportunity cost of the leg capacities consumed by an itinerary request more accurately. Adelman (2006) uses the linear programming representation of the dynamic programming formulation of the network revenue management problem to compute bid-prices. His approach is related to our method in the sense that it computes bid-prices that depend on how much time is left until the departure. Topaloglu (2006) uses a relaxation strategy to decompose the network revenue management problem by the flight legs. His approach is more computationally intensive than our method since it requires solving many network revenue management problems with one-dimensional state variables. Computational experiments indicate that the policies proposed by Talluri and van Ryzin (1999), Bertsimas and Popescu (2003), Adelman (2006) and Topaloglu (2006) perform better than the bid-prices obtained by the deterministic linear program, but these policies are not compared with each other. Finally, other methods, besides bid-prices, have been proposed for solving network revenue management problems. We do not go into the details of these methods and refer the reader to Talluri and van Ryzin (2004) for a comprehensive coverage of the network revenue management field.

We make the following research contributions in this paper. 1) We propose a new method for computing bid-prices. Our method provides bid-prices that depend how much time is left until the departure and partially incorporates the temporal dynamics of the arrivals of the itinerary requests. 2) We show that our method provides an upper bound on the optimal objective value of the network revenue management problem and the upper bound obtained by our method is tighter than the one obtained by the deterministic linear program. 3) We show that the bid-prices obtained by our method are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. 4) We apply our method on network revenue management problems with cancellations. This formally shows how the deterministic linear program should be modified to incorporate cancellations. 5) Computational experiments indicate that the bid-prices obtained by our method perform noticeably better than the ones obtained by the deterministic linear program. Furthermore, our method improves on the upper bounds obtained by many other solution methods. Finally, we compare our method with the policies proposed by Talluri and van Ryzin (1999), Bertsimas and Popescu (2003) and Topaloglu (2006).

The rest of the paper is organized as follows. In Section 1, we formulate the network revenue management problem as a dynamic program. Section 2 presents the fundamental Lagrangian relaxation idea. In Section 3, we compare our method with the deterministic linear program. Section 4 describes the bid-price structure of the policies obtained by our method. Section 5 shows that the bid-prices obtained by our method are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. In Section 6, we apply our method on network revenue management problems with cancellations. Section 7 presents computational experiments.

1 Problem Formulation

We have a set of flight legs that can be used to satisfy the itinerary requests that arrive randomly over time. Whenever an itinerary request arrives, we have to decide whether to accept or reject it. An accepted itinerary request generates a revenue and consumes the capacities on the relevant flight legs. A rejected itinerary request simply leaves the system.

The problem takes place over the finite planning horizon $\mathcal{T} = \{1, \ldots, \tau\}$ and all flight legs depart at time period $\tau + 1$. The set of flight legs is $\mathcal{L}$ and the set of itineraries is $\mathcal{J}$. The capacity on flight leg i is c_i. If a request for itinerary j is accepted, then we generate a revenue of f_j and consume a_{ij} units of capacity on flight leg i. If flight leg i is not in itinerary j, then we have $a_{ij} = 0$. The probability that a request for itinerary j arrives at time period t is p_{jt}. For notational brevity, we assume that $\sum_{j \in \mathcal{J}} p_{jt} = 1$. If there is a positive probability that no itinerary requests arrive at time period t, then we can cover this case by defining a fictitious itinerary j_ϕ with $f_{j_\phi} = 0$ and $p_{j_\phi t} = 1 - \sum_{j \in \mathcal{J}} p_{jt}$.

We let x_{it} be the remaining capacity on flight leg i at time period t so that $x_t = \{x_{it} : i \in \mathcal{L}\}$ gives the state of the system at time period t. We capture the decisions at time period t by $y_t = \{y_{jt} : j \in \mathcal{J}\}$, where y_{jt} takes value 1 if a request for itinerary j is accepted at time period t, and 0 otherwise. Letting e_i be the $|\mathcal{L}|$-dimensional unit vector with a 1 in the element corresponding to $i \in \mathcal{L}$, the optimal policy can be found by computing the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ through the optimality equation

$$V_t(x_t) = \max \quad \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j \, y_{jt} + V_{t+1}(x_t - y_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, e_i) \Big] \tag{1}$$

$$\text{subject to} \quad a_{ij} \, y_{jt} \leq x_{it} \qquad \text{for all } i \in \mathcal{L}, \; j \in \mathcal{J} \tag{2}$$

$$y_{jt} \in \{0, 1\} \qquad \text{for all } j \in \mathcal{J} \tag{3}$$

with the boundary condition that $V_{\tau+1}(\cdot) = 0$. Given the state variable x_t, it is easy to see that the optimal decisions at time period t are given by $\hat{y}_t(x_t) = \{\hat{y}_{jt}(x_t) : j \in \mathcal{J}\}$, where

$$\hat{y}_{jt}(x_t) = \begin{cases} 1 & \text{if } f_j + V_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \geq V_{t+1}(x_t) \text{ and } a_{ij} \leq x_{it} \text{ for all } i \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

A complicating factor in the optimality equation in (1)-(3) is constraints (2). In particular, if these constraints did not exist, then the optimality equation in (1)-(3) would decompose by the time periods. This suggests relaxing these constraints by associating the positive Lagrange multipliers $\{\lambda_{ijt} : i \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}\}$ with them, in which case the optimality equation in (1)-(3) has a simple solution. We make these ideas more precise in the following section.

2 Lagrangian Relaxation Strategy

Associating the positive Lagrange multipliers $\lambda = \{\lambda_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ with constraints (2), we propose solving the optimality equation

$$V_t^\lambda(x_t) = \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ijt} \right] y_{jt} + \sum_{i \in \mathcal{L}} \lambda_{ijt}\, x_{it} + V_{t+1}^\lambda(x_t - y_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) \right\} \right\}, \quad (5)$$

where we scale the Lagrange multipliers by $\{p_{jt} : j \in \mathcal{J}\}$ for notational brevity. If we have $p_{jt} = 0$, then the Lagrange multipliers $\{\lambda_{ijt} : i \in \mathcal{L}\}$ are inconsequential and scaling the Lagrange multipliers in this fashion does not create a problem. We use the superscript λ in the value functions above to emphasize that the solution to the optimality equation in (5) depends on the Lagrange multipliers. The next proposition shows that there is a simple solution to this optimality equation. In the next proposition and throughout the rest of the paper, we let

$$r_{it}^\lambda = \sum_{j \in \mathcal{J}} p_{jt}\, \lambda_{ijt} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau}\, \lambda_{ij\tau}$$

with the boundary condition that $r_{i,\tau+1}^\lambda = 0$.

Proposition 1 *Letting* $[\cdot]^+ = \max\{0, \cdot\}$, *we have*

$$V_t^\lambda(x_t) = \sum_{i \in \mathcal{L}} r_{it}^\lambda\, x_{it} + \sum_{j \in \mathcal{J}} p_{jt} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ijt} - \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,t+1}^\lambda \right]^+ + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ij\tau} - \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,\tau+1}^\lambda \right]^+. \quad (6)$$

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$, (5) implies that

$$V_t^\lambda(x_t) = \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ijt} \right] y_{jt} + \sum_{i \in \mathcal{L}} \lambda_{ijt}\, x_{it} + \sum_{i \in \mathcal{L}} r_{i,t+1}^\lambda \left[x_{it} - a_{ij}\, y_{jt} \right] \right\} \right\} + \sum_{j \in \mathcal{J}} p_{j,t+1} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ij,t+1} - \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,t+2}^\lambda \right]^+ + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ij\tau} - \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,\tau+1}^\lambda \right]^+.$$

Letting $\mathbf{1}(\cdot)$ be the indicator function, the optimal values of the decision variables $\{y_{jt} : j \in \mathcal{J}\}$ in the problem above are $\{\mathbf{1}(f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ijt} - \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,t+1}^\lambda \geq 0) : j \in \mathcal{J}\}$ and the result follows. $\Box$

The next proposition shows that we obtain an upper bound on the value function by solving the optimality equation in (5).

Proposition 2 *If the Lagrange multipliers are positive, then we have* $V_t(x_t) \leq V_t^\lambda(x_t)$.

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$ and letting $\hat{y}_t = \{\hat{y}_{jt} : j \in \mathcal{J}\}$ be the optimal solution to problem (1)-(3), we have

$$V_t^\lambda(x_t) \geq \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \lambda_{ijt} \Big]\, y_{jt} + \sum_{i \in \mathcal{L}} \lambda_{ijt}\, x_{it} + V_{t+1}(x_t - y_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) \Big\} \right\}$$
$$\geq \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j\, \hat{y}_{jt} + V_{t+1}(x_t - \hat{y}_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) \Big] = V_t(x_t),$$

where the first inequality follows by (5) and the induction hypothesis, and the second inequality follows by the nonnegativity of the Lagrange multipliers and the fact that $\hat{y}_t \in \{0,1\}^{|\mathcal{J}|}$ and $a_{ij}\, \hat{y}_{jt} \leq x_{it}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$. □

Since the initial leg capacities are given by $c = \{c_i : i \in \mathcal{L}\}$, the maximum expected revenue over the whole planning horizon is $V_1(c)$. Proposition 2 implies that $V_1(c)$ is bounded from above by $V_1^\lambda(c)$ as long as the Lagrange multipliers are positive. Therefore, to obtain the tightest possible upper bound on $V_1(c)$, we solve the problem

$$\min_{\lambda \geq 0} \left\{ V_1^\lambda(c) \right\}. \tag{7}$$

Since the function $[\cdot]^+$ is convex and $\{r_{it}^\lambda : i \in \mathcal{L},\ t \in \mathcal{T}\}$ are linear functions of the Lagrange multipliers, (6) implies that the objective function of problem (7) is convex. In particular, using the fact that the function $\mathbf{1}(\cdot \geq 0)$ gives a subgradient of the function $[\cdot]^+$ that satisfies $[\ell]^+ \geq [k]^+ + \mathbf{1}(k \geq 0)\, [\ell - k]$, we can obtain a subgradient of $V_1^\lambda(c)$ by simple algebraic manipulations on (6). In this case, problem (7) can be solved by using either subgradient optimization or Benders decomposition; see Wolsey (1998) and Ruszczynski (2003).

3 Comparisons with the Deterministic Linear Program

An alternative method for finding good policies for the network revenue management problem described in Section 1 is to use a deterministic linear program. Letting w_j be the number of requests for itinerary j that we plan to accept over the whole planning horizon, this linear program has the form

$$\max \quad \sum_{j \in \mathcal{J}} f_j\, w_j \tag{8}$$
$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_{ij}\, w_j \leq c_i \qquad \text{for all } i \in \mathcal{L} \tag{9}$$
$$0 \leq w_j \leq \sum_{t \in \mathcal{T}} p_{jt} \qquad \text{for all } j \in \mathcal{J}. \tag{10}$$

Constraints (9) ensure that the itinerary requests that we plan to accept do not violate the leg capacities, whereas constraints (10) ensure that the itinerary requests that we plan to accept do not exceed the expected numbers of itinerary requests.

There are two uses of problem (8)-(10). First, letting $\{\hat{\mu}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated constraints (9), we can use $\hat{\mu}_i$ as an estimate of the opportunity cost of a unit of capacity on flight leg i. These opportunity costs are referred to as the bid-prices in the network revenue management vocabulary and they are used to decide whether to accept or reject an itinerary request. The decision rule is that if the revenue from an itinerary request exceeds the sum of the bid-prices of the flight legs in the requested itinerary, then we accept the itinerary request subject to the capacity availability. Specifically, if we have

$$f_j \geq \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \tag{11}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j. Letting $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\mu}_i \, x_{it}$ for all $t \in \mathcal{T}$, since we have $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} \, e_i) = \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i$, the decision rule in (11) is equivalent to approximating the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ in (4) by $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$. This approach is simple to implement and provides good solutions; see Williamson (1992).

Second, it is possible to show that the optimal objective value of problem (8)-(10) provides an upper bound on the maximum expected revenue over the whole planning horizon. In particular, letting $\hat{\zeta}$ be the optimal objective value of problem (8)-(10), we have $V_1(c) \leq \hat{\zeta}$. The next proposition shows that $V_1(c) \leq \min_{\lambda \geq 0} \{V_1^{\lambda}(c)\} \leq \hat{\zeta}$. Therefore, we can obtain a tighter upper bound on the maximum expected revenue over the whole planning horizon by solving problem (7).

Proposition 3 *We have* $V_1(c) \leq \min_{\lambda \geq 0} \{V_1^{\lambda}(c)\} \leq \hat{\zeta}$.

Proof Since the first inequality follows from Proposition 2, we only show the second inequality. By duality theory, there exist positive Lagrange multipliers $\{\hat{\mu}_i : i \in \mathcal{L}\}$ such that the problem

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{L}} \hat{\mu}_i \, c_i + \sum_{j \in \mathcal{J}} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \Big] w_j \\ \text{subject to} \quad & 0 \leq w_j \leq \sum_{t \in \mathcal{T}} p_{jt} \qquad \text{for all } j \in \mathcal{J} \end{aligned}$$

has the same optimal objective value as problem (8)-(10). The optimal values of the decision variables $\{w_j : j \in \mathcal{J}\}$ in the problem above are $\{\mathbf{1}(f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \geq 0) \sum_{t \in \mathcal{T}} p_{jt} : j \in \mathcal{J}\}$, which implies that

$$\hat{\zeta} = \sum_{i \in \mathcal{L}} \hat{\mu}_i \, c_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \Big]^+ p_{jt}.$$

On the other hand, we define the Lagrange multipliers $\tilde{\lambda} = \{\tilde{\lambda}_{ijt} : i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T}\}$ as $\tilde{\lambda}_{ijt} = 0$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$, $t \in \mathcal{T} \setminus \{\tau\}$ and $\tilde{\lambda}_{ij\tau} = \hat{\mu}_i$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$. In this case, we have $r_{it}^{\tilde{\lambda}} = \hat{\mu}_i$ for all $i \in \mathcal{L}$, $t \in \mathcal{T}$ and (6) implies that

$$V_1^{\tilde{\lambda}}(c) = \sum_{i \in \mathcal{L}} \hat{\mu}_i \, c_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \Big]^+ p_{jt}.$$

Therefore, we have $\min_{\lambda \geq 0} \{V_1^{\lambda}(c)\} \leq V_1^{\tilde{\lambda}}(c) = \hat{\zeta}$. □

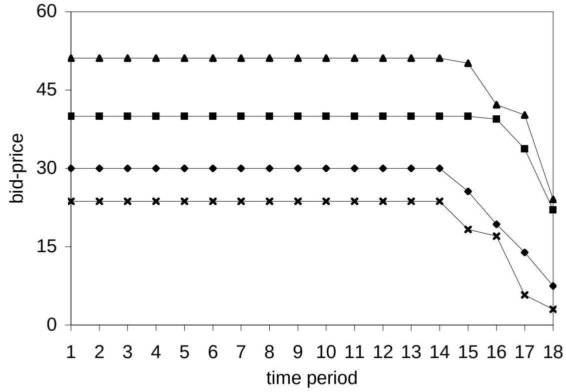

Figure 1: The bid-prices of four flight legs as a function of time.

4 Structure of the Greedy Decision Rule

Letting $\hat{\lambda}$ be the optimal solution to problem (7), we propose approximating the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ in (4) by $\{V_t^{\hat{\lambda}}(\cdot) : t \in \mathcal{T}\}$. Specifically, if we have $f_j + V_{t+1}^{\hat{\lambda}}(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i) \geq V_{t+1}^{\hat{\lambda}}(x_t)$ and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. Since we have $V_{t+1}^{\hat{\lambda}}(x_t) - V_{t+1}^{\hat{\lambda}}(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i) = \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,t+1}^{\hat{\lambda}}$ by (6), this decision rule is equivalent to accepting a request for itinerary j at time period t when we have

$$f_j \geq \sum_{i \in \mathcal{L}} a_{ij}\, r_{i,t+1}^{\hat{\lambda}} \tag{12}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$. The form of the decision rule in (12) is very similar to the one in (11), but the bid-price of a flight leg in (11) is constant, whereas the bid-price of a flight leg in (12) depends on how much time is left until the departure.

For a small test problem with $\tau = 18$, Figure 1 plots $\{r_{it}^{\hat{\lambda}} : t \in \mathcal{T}\}$ for four flight legs. The likelihood of utilizing the capacity on a flight leg diminishes as we approach the end of the planning horizon and the opportunity cost of a unit of capacity decreases. On the other hand, the capacities on the flight legs are plentiful at the beginning of the planning horizon and the opportunity cost of a unit of capacity is constant during this time.

5 Asymptotic Analysis of the Bid-Prices

We consider a family of network revenue management problems $\{P^n : n \in \mathbb{Z}_+\}$ parameterized by the scalar parameter n. Problem P^n takes place over the finite planning horizon $\mathcal{T}^n = \{1, \ldots, n\tau\}$. The capacity on flight leg i in problem P^n is nc_i. Letting $\lceil \cdot \rceil$ be the round up function and $p_{jt}^n = p_{j \lceil t/n \rceil}$, the probability that a request for itinerary j arrives at time period t in problem P^n is p_{jt}^n.

With these definitions, we note that the problem described in Section 1 is P^1. The itinerary arrival probabilities at time periods $\{n(t-1)+1, \ldots, nt\}$ in problem P^n are the same as the itinerary arrival probabilities at time period t in problem P^1. Since we have $\sum_{t \in \mathcal{T}^n} p_{jt}^n = n \sum_{t \in \mathcal{T}^1} p_{jt}^1$, the leg capacities and the expected numbers of itinerary requests in problem P^n are n times larger than those in problem P^1. This is a standard approach in the network revenue management literature to scale the problem to show asymptotic optimality results; see Gallego and van Ryzin (1994) and Talluri and van Ryzin (1998). Our goal is to show that our Lagrangian relaxation strategy provides an asymptotically optimal decision rule for problem P^n as n approaches infinity.

We let $\{V_t(\cdot \,|\, n) : t \in \mathcal{T}^n\}$ be the solution to the optimality equation in (1)-(3) and $\{V_t^{\lambda^n}(\cdot \,|\, n) : t \in \mathcal{T}^n\}$ be the solution to the optimality equation in (5) for problem P^n. These quantities can be obtained by replacing τ with $n\tau$, c_i with nc_i and p_{jt} with p_{jt}^n in the corresponding optimality equations. We note that the Lagrange multipliers for problem P^n are $\lambda^n = \{\lambda_{ijt}^n : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}^n\}$. Letting

$$r_{it}^{\lambda^n} = \sum_{j \in \mathcal{J}} p_{jt}^n \, \lambda_{ijt}^n + \ldots + \sum_{j \in \mathcal{J}} p_{j,n\tau}^n \, \lambda_{ij,n\tau}^n$$

with the boundary condition that $r_{i,n\tau+1}^{\lambda^n} = 0$, Proposition 1 implies that

$$V_1^{\lambda^n}(nc \,|\, n) = n \sum_{i \in \mathcal{L}} r_{i1}^{\lambda^n} \, c_i + \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} p_{jt}^n \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ijt}^n - \sum_{i \in \mathcal{L}} a_{ij} \, r_{i,t+1}^{\lambda^n} \Big]^+.$$

Therefore, the optimal solution to the problem $\min_{\lambda^n \geq 0} \{V_1^{\lambda^n}(nc \,|\, n)\}$ can be found by solving the linear program

$$\begin{aligned}
\min \quad & n \sum_{i \in \mathcal{L}} c_i \, \rho_{i1}^n + \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} p_{jt}^n \, \eta_{jt}^n && && (13)\\
\text{subject to} \quad & \eta_{jt}^n + \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ijt}^n + \sum_{i \in \mathcal{L}} a_{ij} \, \rho_{i,t+1}^n \geq f_j && \text{for all } j \in \mathcal{J},\, t \in \mathcal{T}^n \setminus \{n\tau\} && (14)\\
& \eta_{j,n\tau}^n + \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ij,n\tau}^n \geq f_j && \text{for all } j \in \mathcal{J} && (15)\\
& \rho_{it}^n - \sum_{j \in \mathcal{J}} p_{jt}^n \, \lambda_{ijt}^n - \ldots - \sum_{j \in \mathcal{J}} p_{j,n\tau}^n \, \lambda_{ij,n\tau}^n = 0 && \text{for all } i \in \mathcal{L},\ t \in \mathcal{T}^n && (16)\\
& \eta_{jt}^n, \lambda_{ijt}^n, \rho_{it}^n \geq 0 && \text{for all } i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}^n. && (17)
\end{aligned}$$

Specifically, if we let $\hat{\eta}^n = \{\hat{\eta}_{jt}^n : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$, $\hat{\lambda}^n = \{\hat{\lambda}_{ijt}^n : i \in \mathcal{L}\ , j \in \mathcal{J}\ , t \in \mathcal{T}^n\}$ and $\hat{\rho}^n = \{\hat{\rho}_{it}^n : i \in \mathcal{L},\ t \in \mathcal{T}^n\}$ be the optimal solution to problem (13)-(17), then we have

$$\min_{\lambda^n \geq 0} \{V_1^{\lambda^n}(nc \,|\, n)\} = V_1^{\hat{\lambda}^n}(nc \,|\, n) = n \sum_{i \in \mathcal{L}} c_i \, \hat{\rho}_{i1}^n + \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} p_{jt}^n \, \hat{\eta}_{jt}^n. \tag{18}$$

Letting $\hat{\psi}^n = \{\hat{\psi}_{jt}^n : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$ be the optimal values of the dual variables associated with constraints (14) and (15), and using the boundary condition that $\hat{\rho}_{i,n\tau+1}^n = 0$, we consider the following decision rule for problem P^n. If we have $f_j > \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}_{ijt}^n + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}_{i,t+1}^n$ and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. If, on the other hand, we have $f_j = \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}_{ijt}^n + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}_{i,t+1}^n$ and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t with probability $\hat{\psi}_{jt}^n / p_{jt}^n$. Otherwise, we reject a request for itinerary j at time period t. We refer to this decision rule as the time-dependent bid-price decision rule. The dual of problem (13)-(17) includes the constraints $0 \leq \hat{\psi}_{jt}^n \leq p_{jt}^n$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}^n$ and we can indeed use the quantity

$\hat{\psi}^n_{jt}/p^n_{jt}$ as a probability. As a function of $\hat{\lambda}^n$, $\hat{\rho}^n$ and $\hat{\psi}^n$, we let $B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)$ be the expected revenue obtained over the whole planning horizon by the time-dependent bid-price decision rule for problem P^n. In this section, we show that $\lim_{n\to\infty} B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1(nc\,|\,n) = 1$. Therefore, the expected revenue obtained over the whole planning horizon by the time-dependent bid-price decision rule is asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate.

The next lemma shows important properties of the optimal solution to problem (13)-(17). Its proof is in the appendix.

Lemma 4 *We have*

$$V_1^{\hat{\lambda}^n}(nc\,|\,n) = \sum_{t\in\mathcal{T}^n}\sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1})\,p^n_{jt} + \displaystyle\sum_{t\in\mathcal{T}^n}\sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j = \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1})\,\hat{\psi}^n_{jt} \qquad (19)$$

$$nc_i \geq \sum_{t\in\mathcal{T}^n}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1})\,p^n_{jt} + \displaystyle\sum_{t\in\mathcal{T}^n}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbf{1}(f_j = \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1})\,\hat{\psi}^n_{jt}. \qquad (20)$$

The next lemma shows that $V_1^{\hat{\lambda}^n}(nc\,|\,n)$ increases at least linearly with n. Its proof is in the appendix.

Lemma 5 *We have* $n\,V_1^{\hat{\lambda}^1}(c\,|\,1) \leq V_1^{\hat{\lambda}^n}(nc\,|\,n)$.

The next proposition shows that $\lim_{n\to\infty} B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1(nc\,|\,n) = 1$. Our proof follows from an argument similar to the proof of Theorem 1 in Talluri and van Ryzin (1998). However, we do not smooth the problem by assuming that the revenues from accepting the itinerary requests are continuous random variables. Instead, we directly work with the optimality conditions of a nonsmooth problem.

Proposition 6 *We have* $\lim_{n\to\infty} B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1(nc\,|\,n) = 1$.

Proof We have $B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1^{\hat{\lambda}^n}(nc\,|\,n) \leq B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1(nc\,|\,n) \leq 1$, where the first inequality follows from Proposition 2 and the second inequality follows from the fact that $V_1(nc\,|\,n)$ is the maximum expected revenue over the whole planning horizon for problem P^n. Therefore, it is enough to show that $\lim_{n\to\infty} B(nc\,|\,n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)/V_1^{\hat{\lambda}^n}(nc\,|\,n) \geq 1$.

Letting $f_\phi = \max_{j\in\mathcal{J}} f_j$, we consider a variant of the time-dependent bid-price decision rule for a variant of problem P^n. In the new decision rule, if we have $f_j > \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1}$, then we accept a request for itinerary j at time period t and collect a revenue of f_j, irrespective of the capacity availability. If we have $f_j = \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\lambda}^n_{ijt} + \sum_{i\in\mathcal{L}} a_{ij}\,\hat{\rho}^n_{i,t+1}$, then we accept a request for itinerary j at time period t with probability $\hat{\psi}^n_{jt}/p^n_{jt}$ and collect a revenue of f_j, irrespective of the capacity availability.

If we have $f_j < \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}$, then we reject a request for itinerary j at time period t. In the new decision rule, however, we incur a cost of f_ϕ for each unit of leg capacity sold in excess of the leg capacities $\{ nc_i : i \in \mathcal{L} \}$.

Following the proof of Theorem 1 in Talluri and van Ryzin (1998), one can see that the new decision rule incurs a catastrophic cost whenever it accepts an itinerary request that violates the capacity availability. Furthermore, accepting this itinerary request leaves the system with even less capacity. Therefore, the expected revenue obtained over the whole planning horizon by the new decision rule is smaller than the expected revenue obtained over the whole planning horizon by the time-dependent bid-price decision rule. Specifically, if we let $N(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)$ be the expected revenue obtained over the whole planning horizon by the new decision rule for problem P^n, then we have $N(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n) \leq B(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)$.

We let A^n_{jt} be the number of requests for itinerary j accepted at time period t by the new decision rule for problem P^n. If we have $f_j > \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}$, then $\mathbb{E}\{ A^n_{jt} \} = p^n_{jt}$, whereas if we have $f_j = \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}$, then $\mathbb{E}\{ A^n_{jt} \} = p^n_{jt} \, [\hat{\psi}^n_{jt} / p^n_{jt}] = \hat{\psi}^n_{jt}$. Otherwise, we have $\mathbb{E}\{ A^n_{jt} \} = 0$. Therefore, we obtain

$$
\begin{aligned}
N(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n) &= \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} f_j \, \mathbb{E}\{ A^n_{jt} \} - f_\phi \sum_{i \in \mathcal{L}} \mathbb{E}\{ \big[\textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} - nc_i \big]^+ \} \\
&= \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} f_j \, \mathbf{1}(f_j > \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}) \, p^n_{jt} \\
&\qquad + \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} f_j \, \mathbf{1}(f_j = \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}) \, \hat{\psi}^n_{jt} \\
&\qquad\qquad - f_\phi \sum_{i \in \mathcal{L}} \mathbb{E}\{ \big[\textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} - nc_i \big]^+ \} \\
&= V_1^{\hat{\lambda}^n}(nc \,|\, n) - f_\phi \sum_{i \in \mathcal{L}} \mathbb{E}\{ \big[\textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} - nc_i \big]^+ \} \\
&\leq B(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n),
\end{aligned} \tag{21}
$$

where the third equality follows by (19) and the inequality follows by the fact that $N(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n) \leq B(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)$.

For a random variable Z with finite first two moments, Talluri and van Ryzin (1998) show that $\mathbb{E}\{ [Z - z]^+ \} \leq \sqrt{Var\{Z\}}/2$ for all $z \geq \mathbb{E}\{Z\}$. We have

$$
\begin{aligned}
\mathbb{E}\{ \textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} \} &= \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, \mathbf{1}(f_j > \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}) \, p^n_{jt} \\
&\qquad + \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, \mathbf{1}(f_j = \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}) \, \hat{\psi}^n_{jt}.
\end{aligned}
$$

Therefore, we have $\mathbb{E}\{ \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} \} \leq nc_i$ by (20) and we obtain

$$
\begin{aligned}
\mathbb{E}\{ \big[\textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} - nc_i \big]^+ \} &\leq \frac{1}{2} \sqrt{Var\{ \textstyle\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} \}} \\
&= \frac{1}{2} \sqrt{\textstyle\sum_{t \in \mathcal{T}^n} Var\{ \sum_{j \in \mathcal{J}} a_{ij} \, A^n_{jt} \}} \leq \frac{\sqrt{n \tau \Omega}}{2},
\end{aligned}
$$

where we let $\Omega = \max_{i \in \mathcal{L},\, t \in \mathcal{T}^n} Var\big\{ \sum_{j \in \mathcal{J}} a_{ij}\, A^n_{jt} \big\}$. Using this relationship in (21), we obtain

$$V_1^{\hat{\lambda}^n}(nc \,|\, n) - f_\phi \, |\mathcal{L}| \frac{\sqrt{n \tau \Omega}}{2}$$
$$\leq V_1^{\hat{\lambda}^n}(nc \,|\, n) - f_\phi \sum_{i \in \mathcal{L}} \mathbb{E}\big\{ \big[\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij}\, A^n_{jt} - nc_i \big]^+ \big\} \leq B(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n).$$

Therefore, Lemma 5 implies that

$$1 - f_\phi \, |\mathcal{L}| \frac{\sqrt{n \tau \Omega}}{2n\, V_1^{\hat{\lambda}^1}(c \,|\, 1)} \leq 1 - f_\phi \, |\mathcal{L}| \frac{\sqrt{n \tau \Omega}}{2\, V_1^{\hat{\lambda}^n}(nc \,|\, n)} \leq \frac{B(nc \,|\, n, \hat{\lambda}^n, \hat{\rho}^n, \hat{\psi}^n)}{V_1^{\hat{\lambda}^n}(nc \,|\, n)}.$$

The result follows by taking the limits in the expression above. □

We emphasize that the time-dependent bid-price decision rule analyzed in this section differs from the greedy decision rule described in Section 4 in two ways. First, the greedy decision rule for problem P^1 uses $\big\{ r^{\hat{\lambda}^1}_{i,t+1} : i \in \mathcal{L},\, t \in \mathcal{T} \big\}$ as the bid-prices, whereas the time-dependent bid-price decision rule for problem P^1 uses $\big\{ \hat{\lambda}^1_{ijt} + r^{\hat{\lambda}^1}_{i,t+1} : i \in \mathcal{L},\, j \in \mathcal{J},\; t \in \mathcal{T} \big\}$ as the bid-prices. In our computational experiments, we often have $\hat{\lambda}^1_{ijt} = 0$ for all time periods except for a few time periods at the end of the planning horizon. This implies that $r^{\hat{\lambda}^1}_{i,t+1}$ and $\hat{\lambda}^1_{ijt} + r^{\hat{\lambda}^1}_{i,t+1}$ are equal to each other during a major portion of the planning horizon and the discrepancy between the bid-prices used by the greedy decision rule and the time-dependent bid-price decision rule is not a major concern. Second, the time-dependent bid-price decision rule breaks the tie by using randomization when we have $f_j = \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\lambda}^1_{ijt} + \sum_{i \in \mathcal{L}} a_{ij}\, r^{\hat{\lambda}^1}_{i,t+1}$.

6 Incorporating Cancellations

In this section, we apply the ideas in Sections 2-5 on network revenue management problems with cancellations. In the presence of cancellations, we show that the decision rules obtained by our Lagrangian relaxation strategy are the same as the decision rules obtained by a certain variant of problem (8)-(10). Consequently, our Lagrangian relaxation strategy does not provide a practical advantage for problems with cancellations. Nevertheless, our Lagrangian relaxation strategy formally shows how problem (8)-(10) should be modified to incorporate cancellations. It also gives an alternative proof of the fact that a certain variant of problem (8)-(10) provides an upper bound on the maximum expected revenue over the whole planning horizon.

6.1 Problem Formulation

We use the cancellation model described in Talluri and van Ryzin (2004). The probability that a reservation for itinerary j is retained from time period t to $t+1$ is q_{jt}. That is, the probability that a reservation for itinerary j is canceled at time period t is $1 - q_{jt}$. We assume that the cancellations of different reservations or the cancellations at different time periods are independent and the cancellations at a certain time period occur after the reservations at that time period. Given that there are n_{jt} reservations for itinerary j at time period t, we let $S_{jt}(n_{jt})$ be the number of reservations for itinerary j that we retain from time period t to $t+1$. Due to our assumptions, $S_{jt}(n_{jt})$ has a binomial distribution with parameters (q_{jt}, n_{jt}) and the elements of the vector-valued random variable $S_t(n_t) = \big\{ S_{jt}(n_{jt}) :$

$j \in \mathcal{J}\}$ are independent. If a reservation for itinerary j is denied boarding, then we incur a penalty cost of b_j. For notational brevity, we assume that the cancellations are not refunded.

We let s_{jt} be the number of reservations for itinerary j at time period t so that $s_t = \{s_{jt} : j \in \mathcal{J}\}$ gives the state of the system at time period t. We continue using the decision variables $y_t = \{y_{jt} : j \in \mathcal{J}\}$, where y_{jt} takes value 1 if a reservation for itinerary j is accepted at time period t, and 0 otherwise. We define the decision variables $z = \{z_j : j \in \mathcal{J}\}$, where z_j is the number of reservations for itinerary j that we allow boarding. Letting ϵ_j be the $|\mathcal{J}|$-dimensional unit vector with a 1 in the element corresponding to $j \in \mathcal{J}$, the optimal policy can be found by computing the value functions $\{J_t(\cdot) : t \in \mathcal{T}\}$ through the optimality equation

$$J_t(s_t) = \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j\, y_{jt} + \mathbb{E}\big\{ J_{t+1}(S_t(s_t + y_{jt}\, \epsilon_j)) \big\} \Big] \right\} \tag{22}$$

with the boundary condition that

$$\begin{aligned} J_{\tau+1}(s_{\tau+1}) = \max \quad & - \sum_{j \in \mathcal{J}} b_j \left[s_{j,\tau+1} - z_j \right] && (23) \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} a_{ij}\, z_j \le c_i && \text{for all } i \in \mathcal{L} \quad (24) \\ & z_j \le s_{j,\tau+1} && \text{for all } j \in \mathcal{J} \quad (25) \\ & z_j \in \mathbb{Z}_+ && \text{for all } j \in \mathcal{J}. \quad (26) \end{aligned}$$

We emphasize that the expectation in (22) involves the random variable $S_t(s_t + y_{jt}\, \epsilon_j)$. We solve problem (23)-(26) to decide which reservations should be allowed boarding to minimize the penalty cost. Constraints (24) ensure that our boarding decisions do not violate the leg capacities, whereas constraints (25) ensure that the reservations that we allow boarding do not exceed the reservations that we retain until time period $\tau + 1$.

In the next section, we apply our Lagrangian relaxation strategy on the optimality equation in (22).

6.2 Lagrangian Relaxation Strategy

Associating the positive Lagrange multipliers $\alpha = \{\alpha_i : i \in \mathcal{L}\}$ with constraints (24), we propose solving the optimality equation

$$J^\alpha_t(s_t) = \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j\, y_{jt} + \mathbb{E}\big\{ J^\alpha_{t+1}(S_t(s_t + y_{jt}\, \epsilon_j)) \big\} \Big] \right\} \tag{27}$$

with the boundary condition that

$$\begin{aligned} J^\alpha_{\tau+1}(s_{\tau+1}) = \max \quad & \sum_{j \in \mathcal{J}} \Big[b_j - \sum_{i \in \mathcal{L}} a_{ij}\, \alpha_i \Big] z_j - \sum_{j \in \mathcal{J}} b_j\, s_{j,\tau+1} + \sum_{i \in \mathcal{L}} c_i\, \alpha_i && (28) \\ \text{subject to} \quad & (25), (26). && (29) \end{aligned}$$

The next proposition shows that there is a simple solution to this optimality equation. In the next proposition and throughout the rest of the paper, we let $R_j^\alpha = -\min\{b_j,\ \sum_{i\in\mathcal{L}} a_{ij}\,\alpha_i\}$ and $Q_{jt} = q_{jt}\ldots q_{j\tau}$. With this definition, we note that Q_{jt} is the probability that a reservation for itinerary j at time period t is retained until time period $\tau+1$.

Proposition 7 *We have*

$$J_t^\alpha(s_t) = \sum_{j\in\mathcal{J}} R_j^\alpha\, Q_{jt}\, s_{jt} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i + \sum_{j\in\mathcal{J}} p_{jt}\big[f_j + R_j^\alpha\, Q_{jt}\big]^+ + \ldots + \sum_{j\in\mathcal{J}} p_{j\tau}\big[f_j + R_j^\alpha\, Q_{j\tau}\big]^+. \tag{30}$$

Proof We show the result by induction over the time periods. The optimal values of the decision variables $\{z_j : j\in\mathcal{J}\}$ in problem (28)-(29) are $\{\mathbf{1}(b_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha_i \geq 0)\, s_{j,\tau+1} : j\in\mathcal{J}\}$. Therefore, we have

$$J_{\tau+1}^\alpha(s_{\tau+1}) = \sum_{j\in\mathcal{J}}\Big[b_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha_i\Big]^+ s_{j,\tau+1} - \sum_{j\in\mathcal{J}} b_j\, s_{j,\tau+1} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i = \sum_{j\in\mathcal{J}} R_j^\alpha\, s_{j,\tau+1} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i.$$

Using the expression above and the fact that $S_{j\tau}(n_{j\tau})$ has a binomial distribution with parameters $(q_{j\tau}, n_{j\tau})$, we obtain

$$\begin{aligned}\mathbb{E}\big\{J_{\tau+1}^\alpha(S_\tau(s_\tau + y_{j\tau}\,\epsilon_j))\big\} &= \mathbb{E}\Big\{\sum_{j\in\mathcal{J}} R_j^\alpha\, S_{j\tau}(s_{j\tau}) - R_j^\alpha\, S_{j\tau}(s_{j\tau}) + R_j^\alpha\, S_{j\tau}(s_{j\tau} + y_{j\tau})\Big\} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i \\ &= \sum_{j\in\mathcal{J}} R_j^\alpha\, q_{j\tau}\, s_{j\tau} + R_j^\alpha\, q_{j\tau}\, y_{j\tau} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i.\end{aligned}$$

Therefore, (27) implies that

$$\begin{aligned}J_\tau^\alpha(s_\tau) &= \max_{y_\tau\in\{0,1\}^{|\mathcal{J}|}}\left\{\sum_{j\in\mathcal{J}} p_{j\tau}\big[f_j + R_j^\alpha\, q_{j\tau}\big] y_{j\tau}\right\} + \sum_{j\in\mathcal{J}} R_j^\alpha\, q_{j\tau}\, s_{j\tau} + \sum_{i\in\mathcal{L}} \alpha_i\, c_i \\ &= \sum_{j\in\mathcal{J}} p_{j\tau}\big[f_j + R_j^\alpha\, q_{j\tau}\big]^+ + \sum_{j\in\mathcal{J}} R_j^\alpha\, q_{j\tau}\, s_{j\tau} + \sum_{i\in\mathcal{L}} c_i\,\alpha_i,\end{aligned}$$

which shows that the result holds for time period τ. Assuming that the result holds for time period $t+1$, it is easy to show that the result holds for time period t. □

The next proposition shows that we obtain an upper bound on the value function by solving the optimality equation in (27). Its proof follows the same argument in the proof of Proposition 2.

Proposition 8 *If the Lagrange multipliers are positive, then we have $J_t(s_t) \leq J_t^\alpha(s_t)$.*

Since we do not have any reservations initially, the maximum expected revenue over the whole planning horizon is $J_1(0)$. Proposition 8 implies that $J_1(0)$ is bounded from above by $J_1^\alpha(0)$ as long as the Lagrange multipliers are positive. Therefore, to obtain the tightest possible upper bound on $J_1(0)$, we solve the problem

$$\min_{\alpha\geq 0}\Big\{J_1^\alpha(0)\Big\}. \tag{31}$$

Since the function $[\cdot]^+$ is convex and increasing, and $\{R_j^\alpha : j \in \mathcal{J}\}$ are convex functions of the Lagrange multipliers, (30) implies that the objective function of problem (31) is convex. In this case, we can obtain a subgradient of $J_1^\alpha(0)$ by simple algebraic manipulations on (30) and easily solve problem (31).

6.3 Comparisons with the Deterministic Linear Program

This section relates problem (31) to a certain variant of problem (8)-(10). Letting u_{jt} be the number of reservations for itinerary j that we plan to accept at time period t and v_j be the number of reservations for itinerary j that we plan to deny boarding, this variant of problem (8)-(10) has the form

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \, u_{jt} - \sum_{j \in \mathcal{J}} b_j \, v_j \tag{32}$$

$$\text{subject to} \quad \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} a_{ij} \, Q_{jt} \, u_{jt} - \sum_{j \in \mathcal{J}} a_{ij} \, v_j \leq c_i \qquad \text{for all } i \in \mathcal{L} \tag{33}$$

$$\sum_{t \in \mathcal{T}} Q_{jt} \, u_{jt} - v_j \geq 0 \qquad \text{for all } j \in \mathcal{J} \tag{34}$$

$$0 \leq u_{jt} \leq p_{jt} \qquad \text{for all } j \in \mathcal{J}, \; t \in \mathcal{T} \tag{35}$$

$$v_j \geq 0 \qquad \text{for all } j \in \mathcal{J}. \tag{36}$$

Since Q_{jt} is the probability that a reservation for itinerary j at time period t is retained until time period $\tau + 1$, the first term on the left side of constraints (33) accounts for the leg capacities consumed by the reservations that we plan to accept and expect to retain until time period $\tau + 1$. Constraints (34) ensure that the reservations that we plan to deny boarding do not exceed the reservations that we plan to accept and expect to retain until time period $\tau + 1$. Constraints (35) ensure that the reservations that we plan to accept do not exceed the expected numbers of reservations.

Letting $\hat{\xi}$ be the optimal objective value of problem (32)-(36), the next proposition shows that we have $\min_{\alpha \geq 0} \{J_1^\alpha(0)\} = \hat{\xi}$.

Proposition 9 *We have* $\min_{\alpha \geq 0} \{J_1^\alpha(0)\} = \hat{\xi}$.

Proof Associating the positive Lagrange multipliers $\{\alpha_i : i \in \mathcal{L}\}$ with constraints (33), we let $\hat{\xi}^\alpha$ be the optimal objective value of the problem

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{L}} c_i \, \alpha_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, Q_{jt} \, \alpha_i \Big] u_{jt} + \sum_{j \in \mathcal{J}} \Big[\sum_{i \in \mathcal{L}} a_{ij} \, \alpha_i - b_j \Big] v_j \\ \text{subject to} \quad & (34), (35), (36). \end{aligned}$$

We have $\hat{\xi} = \min_{\alpha \geq 0} \{\hat{\xi}^\alpha\}$ by duality theory. The optimal values of the decision variables $\{v_j : j \in \mathcal{J}\}$ in the problem above are $\{\mathbf{1}(\sum_{i \in \mathcal{L}} a_{ij} \, \alpha_i - b_j \geq 0) \sum_{t \in \mathcal{T}} Q_{jt} \, u_{jt} : j \in \mathcal{J}\}$, in which case we can drop these decision variables by plugging their optimal values into the objective function. Therefore, the optimal objective value of the problem

$$\max \quad \sum_{i \in \mathcal{L}} c_i \, \alpha_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \Big\{ f_j - \sum_{i \in \mathcal{L}} a_{ij} \, Q_{jt} \, \alpha_i + \Big[\sum_{i \in \mathcal{L}} a_{ij} \, \alpha_i - b_j \Big]^+ Q_{jt} \Big\} u_{jt} \tag{37}$$

$$\text{subject to} \quad 0 \leq u_{jt} \leq p_{jt} \qquad \text{for all } j \in \mathcal{J}, \; t \in \mathcal{T} \tag{38}$$

is also $\hat{\xi}^\alpha$. Using the definition of R_j^α, the objective function of problem (37)-(38) can be written as

$$\sum_{i\in\mathcal{L}} c_i\,\alpha_i + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}} \big[f_j + R_j^\alpha\, Q_{jt}\big]\, u_{jt}.$$

In this case, the optimal values of the decision variables $\{u_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$ in problem (37)-(38) are $\{\mathbf{1}(f_j + R_j^\alpha\, Q_{jt} \geq 0)\, p_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$ and we obtain

$$\hat{\xi}^\alpha = \sum_{i\in\mathcal{L}} c_i\,\alpha_i + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}} \big[f_j + R_j^\alpha\, Q_{jt}\big]^+ p_{jt}.$$

Therefore, (30) implies that $J_1^\alpha(0) = \hat{\xi}^\alpha$ and the result follows. □

Propositions 8 and 9 imply that $J_1(0) \leq \hat{\xi}$ and the optimal objective value of problem (32)-(36) provides an upper bound on the maximum expected revenue over the whole planning horizon. Furthermore, Proposition 9 formally shows that the deterministic linear program corresponding to the optimality equation in (22) should have the form of problem (32)-(36).

Problem (32)-(36) is the analogue of problem (8)-(10) that incorporates cancellations. Although problem (8)-(10) is ubiquitous in the literature, problem (32)-(36) is rarely mentioned. Bertsimas and Popescu (2003) formulate a similar problem, but they assume that $Q_{j1} = \ldots = Q_{j\tau}$ for all $j \in \mathcal{J}$, which cannot hold when we have $0 < q_{jt} < 1$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$.

7 Computational Experiments

In this section, we compare the performances of numerous solution methods for the network revenue management problem described in Section 1.

7.1 Benchmark Strategies

We compare the performances of the following six solution methods.

• Lagrangian Relaxation Strategy (LR). This is the solution method developed in this paper, but our implementation tries to refine the approximation to the value function at each time period by resolving problem (7). In particular, given the state variable x_t at time period t, we solve the problem $\min_{\lambda\geq 0}\{V_t^\lambda(x_t)\}$ to obtain the optimal solution $\hat{\lambda}_t(x_t)$. If we have $f_j \geq \sum_{i\in\mathcal{L}} a_{ij}\, r_{i,t+1}^{\hat{\lambda}_t(x_t)}$, then we accept a request for itinerary j at time period t subject to the capacity availability.

• Alternative Lagrangian Relaxation Strategy (LR-A). Defining a fictitious flight leg i_ϕ with infinite capacity and the decision variables $\{y_{ijt} : i \in \mathcal{L} \cup \{i_\phi\},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$, LR-A is based on writing the optimality equation in (1) as

$$V_t(x_t) = \max \quad \sum_{j\in\mathcal{J}} p_{jt}\Big[f_j\, y_{i_\phi jt} + V_{t+1}(x_t - \textstyle\sum_{i\in\mathcal{L}} y_{ijt}\, a_{ij}\, e_i)\Big] \tag{39}$$

$$\text{subject to} \quad a_{ij}\, y_{ijt} \leq x_{it} \qquad \text{for all } i \in \mathcal{L},\ j \in \mathcal{J} \tag{40}$$

$$y_{ijt} - y_{i_\phi jt} = 0 \qquad \text{for all } i \in \mathcal{L},\ j \in \mathcal{J} \tag{41}$$

$$y_{ijt} \in \{0,1\} \qquad \text{for all } i \in \mathcal{L} \cup \{i_\phi\},\ j \in \mathcal{J}. \tag{42}$$

In this case, we can associate the Lagrange multipliers $\beta = \{\beta_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ with constraints (41) to obtain the optimality equation

$$
\begin{aligned}
V_t^\beta(x_t) = \max \quad & \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_j - \sum_{i \in \mathcal{L}} \beta_{ijt} \Big] y_{i_\phi jt} + \sum_{i \in \mathcal{L}} \beta_{ijt}\, y_{ijt} + V_{t+1}^\beta(x_t - \textstyle\sum_{i \in \mathcal{L}} y_{ijt}\, a_{ij}\, e_i) \Big\} \\
\text{subject to} \quad & (40), (42).
\end{aligned}
$$

One can show that the optimality equation above decomposes by the flight legs and $V_1^\beta(c)$ can be computed by solving $|\mathcal{L}|$ network revenue management problems, each of which involves a one-dimensional state variable. It is also possible to show that we have $V_1(c) \leq V_1^\beta(c)$. To obtain the tightest possible upper bound on $V_1(c)$, we solve the problem $\min_\beta \{V_1^\beta(c)\}$; see Topaloglu (2006).

Similar to LR, our implementation of LR-A tries to refine the approximation to the value function at each time period. In particular, given the state variable x_t at time period t, we solve the problem $\min_\beta \{V_t^\beta(x_t)\}$ to obtain the optimal solution $\hat{\beta}_t(x_t)$. Following the decision rule in (4), if we have

$$
f_j \geq V_{t+1}^{\hat{\beta}_t(x_t)}(x_t) - V_{t+1}^{\hat{\beta}_t(x_t)}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i), \tag{43}
$$

then we accept a request for itinerary j at time period t subject to the capacity availability. From the computational perspective, LR-A is more expensive than LR because there does not exist a simple expression for $V_t^\beta(x_t)$ comparable to (6).

• Deterministic Linear Program (LP-D). This is the solution method described in Section 3. Given the state variable x_t at time period t, we replace the right side of constraints (9) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (10) with $\{\sum_{t'=t}^{\tau} p_{jt'} : j \in \mathcal{J}\}$, and solve problem (8)-(10). Letting $\{\hat{\mu}_{it}(x_t) : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (9), if we have $f_j \geq \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\mu}_{it}(x_t)$, then we accept a request for itinerary j at time period t subject to the capacity availability; see Talluri and van Ryzin (2004).

• Randomized Linear Program (LP-R). LP-D uses only the expected numbers of itinerary requests and LP-R tries to make up for this deficiency. We let D_{jt} be the number of requests for itinerary j at time period t so that we have $\mathbb{P}\{D_{jt} = 1\} = p_{jt}$ and $\mathbb{P}\{D_{jt} = 0\} = 1 - p_{jt}$. We generate K independent samples of $D = \{D_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$, which we denote by $\tilde{D}^k = \{\tilde{D}_{jt}^k : j \in \mathcal{J},\ t \in \mathcal{T}\}$ for $k = 1, \ldots, K$. Given the state variable x_t at time period t, we replace the right side of constraints (9) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (10) with $\{\sum_{t'=t}^{\tau} \tilde{D}_{jt'}^k : j \in \mathcal{J}\}$, and solve problem (8)-(10). Letting $\hat{L}_t(x_t, \tilde{D}^k)$ be the optimal objective value of this problem and $\{\hat{\mu}_{it}(x_t, \tilde{D}^k) : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (9), if we have

$$
f_j \geq \frac{1}{K} \sum_{k=1}^{K} \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\mu}_{it}(x_t, \tilde{D}^k),
$$

then we accept a request for itinerary j at time period t subject to the capacity availability; see Talluri and van Ryzin (1999). It is also possible to show that $V_1(c) \leq \mathbb{E}\{\hat{L}_1(c, D)\}$. Therefore, LP-R provides an upper bound on the maximum expected revenue over the whole planning horizon, but computing the expectation $\mathbb{E}\{\hat{L}_1(c, D)\}$ requires estimation through simulation.

• Finite Differences on Deterministic Linear Program (FD-D). FD-D tries to capture the total opportunity cost of the leg capacities consumed by an itinerary request more accurately. Given the state variable x_t at time period t, we replace the right side of constraints (9) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (10) with $\{\sum_{t'=t}^{\tau} p_{j't} : j \in \mathcal{J}\}$, and solve problem (8)-(10) to obtain the optimal objective value $\hat{L}_t(x_t)$. We then replace the right side of constraints (9) with $\{x_{it} - a_{ij} : i \in \mathcal{L}\}$ and resolve problem (8)-(10) to obtain the optimal objective value $\hat{L}^-_{jt}(x_t)$. If we have $f_j \geq \hat{L}_t(x_t) - \hat{L}^-_{jt}(x_t)$, then we accept a request for itinerary j at time period t subject to the capacity availability; see Bertsimas and Popescu (2003).

• Finite Differences on Randomized Linear Program (FD-R). FD-R is a natural extension of LP-R and FD-D. We generate K independent samples of $D = \{D_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$, which we denote by $\tilde{D}^k = \{\tilde{D}^k_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$ for $k = 1, \ldots, K$. Given the state variable x_t at time period t, we replace the right side of constraints (9) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (10) with $\{\sum_{t'=t}^{\tau} \tilde{D}^k_{jt'} : j \in \mathcal{J}\}$, and solve problem (8)-(10) to obtain the optimal objective value $\hat{L}_t(x_t, \tilde{D}^k)$. We then replace the right side of constraints (9) with $\{x_{it} - a_{ij} : i \in \mathcal{L}\}$ and resolve problem (8)-(10) to obtain the optimal objective value $\hat{L}^-_{jt}(x_t, \tilde{D}^k)$. If we have

$$f_j \geq \frac{1}{K} \sum_{k=1}^{K} \hat{L}_t(x_t, \tilde{D}^k) - \hat{L}^-_{jt}(x_t, \tilde{D}^k), \tag{44}$$

then we accept a request for itinerary j at time period t subject to the capacity availability. Although FD-R is a natural extension of LP-R and FD-D, it did not previously appear in the literature.

7.2 Experimental Setup

In our test problems, we consider an airline network serving N locations out of a single hub. This is an important network structure that frequently arises in practice. Associated with each location, there are two flight legs, one of which is to the hub and the other is from the hub. There is a high-fare and a low-fare itinerary that connects each origin-destination pair. Consequently, we have $2N$ flight legs and $2N(N+1)$ itineraries, $4N$ of which include one flight leg and $2N(N-1)$ of which include two flight legs. The revenues associated with the high-fare itineraries are κ times larger than the revenues associated with the low-fare itineraries. The probability of having a request for a high-fare itinerary increases over time, whereas the probability of having a request for a low-fare itinerary decreases over time. Since we have $\mathbb{E}\{D_{jt}\} = p_{jt}$, the expected demand for the capacity on flight leg i is $\sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} a_{ij}\, p_{jt}$ and we measure the tightness of the leg capacities by

$$\theta = \frac{\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} a_{ij}\, p_{jt}}{\sum_{i \in \mathcal{L}} c_i}.$$

We vary N, θ and κ in our test problems and label our test problems by the triplet (N, θ, κ). We have $\tau = 100$ in all of our test problems and use $K = 250$ for LP-R and FD-R. To solve the problem $\min_{\lambda \geq 0} \{V^\lambda_t(x_t)\}$ as accurately as possible, we use Benders decomposition instead of subgradient optimization. This avoids the problem of step size and termination criterion selection.

Problem (N, θ, κ)	LR $V_1^{\hat{\lambda}_1(c)}(c)$	LR-A $V_1^{\hat{\beta}_1(c)}(c)$	LP-D $\hat{\zeta}$	LP-R $\mathbb{E}\{\hat{L}_1(c, D)\}$	Rankings	$V_1^{\hat{\lambda}_1(c)}(c)/\hat{\zeta}$	Best/$\hat{\zeta}$	CPU
$(2, 1.0, 2)$	3,857	3,655	3,918	3,738	$(3, 1, 4, 2)$	98.43	93.28	1.52
$(2, 1.0, 4)$	5,441	5,179	5,513	5,332	$(3, 1, 4, 2)$	98.68	93.94	0.73
$(2, 1.0, 8)$	8,630	8,306	8,704	8,521	$(3, 1, 4, 2)$	99.15	95.43	0.52
$(3, 1.0, 2)$	5,864	5,538	5,966	5,665	$(3, 1, 4, 2)$	98.29	92.83	6.41
$(3, 1.0, 4)$	8,360	7,945	8,478	8,175	$(3, 1, 4, 2)$	98.61	93.72	2.95
$(3, 1.0, 8)$	13,383	12,878	13,501	13,196	$(3, 1, 4, 2)$	99.12	95.39	1.94
$(4, 1.0, 2)$	7,320	6,848	7,460	7,023	$(3, 1, 4, 2)$	98.12	91.80	30.03
$(4, 1.0, 4)$	10,521	9,908	10,691	10,248	$(3, 1, 4, 2)$	98.41	92.68	14.03
$(4, 1.0, 8)$	16,978	16,228	17,152	16,697	$(3, 1, 4, 2)$	98.98	94.61	5.63
$(2, 1.2, 2)$	3,384	3,252	3,448	3,395	$(2, 1, 4, 3)$	98.14	94.30	1.61
$(2, 1.2, 4)$	4,966	4,752	5,044	4,989	$(2, 1, 4, 3)$	98.47	94.23	0.91
$(2, 1.2, 8)$	8,155	7,861	8,234	8,178	$(2, 1, 4, 3)$	99.04	95.47	0.59
$(3, 1.2, 2)$	5,231	4,984	5,339	5,194	$(3, 1, 4, 2)$	97.97	93.34	5.53
$(3, 1.2, 4)$	7,727	7,356	7,851	7,704	$(3, 1, 4, 2)$	98.42	93.69	3.47
$(3, 1.2, 8)$	12,750	12,255	12,874	12,724	$(3, 1, 4, 2)$	99.03	95.19	2.19
$(4, 1.2, 2)$	6,509	6,167	6,691	6,467	$(3, 1, 4, 2)$	97.28	92.18	50.74
$(4, 1.2, 4)$	9,702	9,175	9,921	9,691	$(3, 1, 4, 2)$	97.79	92.48	13.50
$(4, 1.2, 8)$	16,156	15,478	16,382	16,141	$(3, 1, 4, 2)$	98.62	94.48	7.50
$(2, 1.6, 2)$	2,716	2,604	2,783	2,775	$(2, 1, 4, 3)$	97.60	93.58	1.36
$(2, 1.6, 4)$	4,297	4,090	4,378	4,369	$(2, 1, 4, 3)$	98.13	93.41	0.69
$(2, 1.6, 8)$	7,485	7,191	7,569	7,556	$(2, 1, 4, 3)$	98.89	95.01	0.52
$(3, 1.6, 2)$	4,367	4,150	4,483	4,398	$(2, 1, 4, 3)$	97.42	92.57	7.88
$(3, 1.6, 4)$	6,857	6,470	6,995	6,908	$(2, 1, 4, 3)$	98.03	92.50	3.75
$(3, 1.6, 8)$	11,880	11,343	12,018	11,927	$(2, 1, 4, 3)$	98.85	94.38	2.61
$(4, 1.6, 2)$	5,216	4,943	5,401	5,306	$(2, 1, 4, 3)$	96.56	91.51	51.58
$(4, 1.6, 4)$	8,406	7,905	8,632	8,525	$(2, 1, 4, 3)$	97.38	91.59	12.58
$(4, 1.6, 8)$	14,858	14,165	15,093	14,965	$(2, 1, 4, 3)$	98.44	93.85	6.33

Table 1: Upper bounds on the maximum expected revenue over the whole planning horizon.

7.3 Computational Results

As described in Sections 2, 3 and 7.1, LR, LR-A, LP-D and LP-R provide upper bounds on the maximum expected revenue over the whole planning horizon. For different test problems, Table 1 compares the upper bounds obtained by these four solution methods. The second, third, fourth and fifth columns in this table respectively show the upper bounds obtained by LR, LR-A, LP-D and LP-R. To compute the upper bound obtained by LP-R, we estimate the expectation $\mathbb{E}\{\hat{L}_1(c, D)\}$ through simulation with $\mp 0.5\%$ accuracy and 95% confidence. The sixth column shows the rankings of the upper bounds obtained by LR, LR-A, LP-D and LP-R. For example, $(3, 1, 4, 2)$ indicates that the upper bounds obtained by LR, LR-A, LP-D and LP-R are respectively the third, first, fourth and second tightest upper bounds. The seventh column shows the ratios of the upper bounds obtained by LR and LP-D, whereas the eighth column shows the ratios of the upper bounds obtained by the best solution method and LP-D. For example, the eighth column for test problem $(2, 1.0, 2)$ shows the ratio of the upper bounds obtained by LR-A and LP-D. LP-D is frequently used to solve network revenue management problems in practice and the purpose of the seventh and eighth columns is to show how much we can improve the upper bounds obtained by LP-D by using other solution methods. The ninth column shows the CPU seconds required to solve problem (7) on a Pentium IV PC running Windows XP with 2.4 GHz of CPU and 1 GB of RAM.

Problem (N, θ, κ)	LR	LR-A	LP-D	LP-R	FD-D	FD-R	Rankings	LR/LP-D	Best/LP-D
$(2, 1.0, 2)$	3,638	3,648	3,637	3,640	3,634	3,643	$(4, 1, 5, 3, 6, 2)$	100.04	100.30
$(2, 1.0, 4)$	5,134	5,178	5,076	5,151	5,120	5,172	$(4, 1, 6, 3, 5, 2)$	101.14	102.01
$(2, 1.0, 8)$	8,208	8,305	7,956	8,239	8,106	8,290	$(4, 1, 6, 3, 5, 2)$	103.17	104.39
$(3, 1.0, 2)$	5,460	5,471	5,456	5,450	5,446	5,467	$(3, 1, 4, 5, 6, 2)$	100.08	100.27
$(3, 1.0, 4)$	7,829	7,896	7,718	7,811	7,795	7,884	$(3, 1, 6, 4, 5, 2)$	101.44	102.31
$(3, 1.0, 8)$	12,695	12,871	12,248	12,723	12,496	12,845	$(4, 1, 6, 3, 5, 2)$	103.65	105.09
$(4, 1.0, 2)$	6,670	6,696	6,659	6,644	6,652	6,698	$(3, 2, 4, 6, 5, 1)$	100.16	100.59
$(4, 1.0, 4)$	9,566	9,654	9,362	9,474	9,507	9,657	$(3, 2, 6, 5, 4, 1)$	102.17	103.15
$(4, 1.0, 8)$	15,496	15,873	14,900	15,622	15,310	15,949	$(4, 2, 6, 3, 5, 1)$	104.00	107.03
$(2, 1.2, 2)$	3,224	3,228	3,210	3,210	3,205	3,214	$(2, 1, 4, 5, 6, 3)$	100.42	100.56
$(2, 1.2, 4)$	4,692	4,721	4,562	4,682	4,643	4,710	$(3, 1, 6, 4, 5, 2)$	102.84	103.48
$(2, 1.2, 8)$	7,698	7,825	7,262	7,743	7,521	7,814	$(4, 1, 6, 3, 5, 2)$	106.01	107.76
$(3, 1.2, 2)$	4,860	4,889	4,833	4,854	4,839	4,886	$(3, 1, 6, 4, 5, 2)$	100.56	101.15
$(3, 1.2, 4)$	7,174	7,272	6,950	7,136	7,105	7,259	$(3, 1, 6, 4, 5, 2)$	103.21	104.62
$(3, 1.2, 8)$	11,855	12,227	11,164	11,977	11,650	12,184	$(4, 1, 6, 3, 5, 2)$	106.18	109.52
$(4, 1.2, 2)$	5,867	5,941	5,861	5,860	5,868	5,938	$(4, 1, 5, 6, 3, 2)$	100.10	101.36
$(4, 1.2, 4)$	8,681	8,875	8,410	8,710	8,675	8,886	$(4, 2, 6, 3, 5, 1)$	103.22	105.65
$(4, 1.2, 8)$	14,556	15,047	13,574	14,701	14,340	15,077	$(4, 2, 6, 3, 5, 1)$	107.23	111.07
$(2, 1.6, 2)$	2,568	2,578	2,550	2,562	2,555	2,569	$(3, 1, 6, 4, 5, 2)$	100.68	101.09
$(2, 1.6, 4)$	4,000	4,055	3,821	4,024	3,960	4,045	$(4, 1, 6, 3, 5, 2)$	104.67	106.11
$(2, 1.6, 8)$	6,880	7,156	6,351	7,053	6,783	7,160	$(4, 2, 6, 3, 5, 1)$	108.33	112.73
$(3, 1.6, 2)$	4,014	4,060	3,994	4,041	4,006	4,050	$(4, 1, 6, 3, 5, 2)$	100.50	101.65
$(3, 1.6, 4)$	6,269	6,378	6,037	6,279	6,226	6,376	$(4, 1, 6, 3, 5, 2)$	103.84	105.65
$(3, 1.6, 8)$	10,818	11,268	10,093	11,105	10,624	11,296	$(4, 2, 6, 3, 5, 1)$	107.18	111.91
$(4, 1.6, 2)$	4,680	4,699	4,640	4,655	4,651	4,736	$(3, 2, 6, 4, 5, 1)$	100.86	102.07
$(4, 1.6, 4)$	7,312	7,482	7,032	7,373	7,313	7,575	$(5, 2, 6, 3, 4, 1)$	103.97	107.72
$(4, 1.6, 8)$	12,721	13,483	11,817	13,369	12,686	13,726	$(4, 2, 6, 3, 5, 1)$	107.65	116.15

Table 2: Expected revenues over the whole planning horizon.

Proposition 3 shows that the upper bound obtained by LR is tighter than the one obtained by LP-D. Similarly, Topaloglu (2006) and Talluri and van Ryzin (1999) show that the upper bounds obtained by LR-A and LP-R are tighter than the upper bound obtained by LP-D. Therefore, it is not surprising that the upper bounds obtained by LR, LR-A and LP-R are tighter than the upper bound obtained by LP-D for all test problems. LR improves the upper bounds obtained by LP-D by up to 3.5% and the best solution method improves the upper bounds obtained by LP-D by up to 8.5%. LR-A consistently provides the tightest upper bounds. For many test problems, LR improves the upper bounds obtained by LP-R and this especially appears to be the case for problems with tight leg capacities.

Table 2 compares the expected revenues obtained over the whole planning horizon by different solution methods. The second, third, fourth, fifth, sixth and seventh columns in this table respectively show the expected revenues obtained by LR, LR-A, LP-D, LP-R, FD-D and FD-R. We estimate all of the expected revenues through simulation with $\mp 0.5\%$ accuracy and 95% confidence. We use common random numbers when simulating the performances of different solution methods under different trajectories of the itinerary arrivals; see Law and Kelton (2000). The eighth column shows the rankings of the expected revenues obtained by LR, LR-A, LP-D, LP-R, FD-D and FD-R. The interpretation of this column is similar to that of the sixth column in Table 1. The ninth column shows the ratios of the expected revenues obtained by LR and LP-D, whereas the tenth column shows the ratios of the

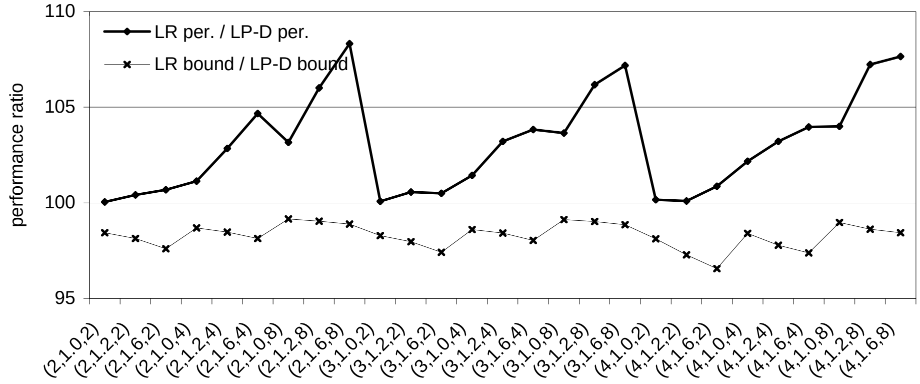

Figure 2: Ratios of the expected revenues and upper bounds obtained by LR and LP-D.

expected revenues obtained by the best solution method and LP-D. For example, the tenth column for test problem $(4, 1.0, 2)$ shows the ratio of the expected revenues obtained by FD-R and LP-D.

The results indicate that LR-A and FD-R consistently provide the highest expected revenues, whereas LP-D and FD-D consistently provide the lowest expected revenues. LR and LP-R compete for the third and fourth places. LR can perform up to 8% better than LP-D and the best solution method can perform up to 16% better than LP-D. For many test problems, the expected revenues obtained by LR are higher than those obtained by LP-D, LP-R and FD-D.

To illustrate how different problem parameters affect the performance gap between LR and LP-D, the thick data series in Figure 2 plot the ratios of the expected revenues obtained by LR and LP-D. In this figure, blocks of three consecutive test problems in the horizontal axis share the same problem characteristics other than the tightness of the leg capacities. Consequently, the saw tooth pattern of the thick data series indicates that tight leg capacities cause the performance gap between LR and LP-D to grow. Similarly, the thin data series in Figure 2 plot the ratios of the upper bounds obtained by LR and LP-D. The saw tooth pattern of the thin data series indicates that tight leg capacities also cause the gap between the upper bounds obtained by LR and LP-D to grow.

8 Conclusions

In this paper, we presented a new method for computing bid-prices in network revenue management problems. Our method provides an upper bound on the optimal objective value of the problem and this upper bound is tighter than the one obtained by the deterministic linear program. We showed that the bid-prices obtained by our method are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. We applied our method on problems with cancellations to formally show how the deterministic linear program should be modified to incorporate cancellations.

Our computational experiments indicated that the expected revenues obtained by LR are consistently higher than those obtained by LP-D and FD-D, which essentially use a deterministic approximation of the network revenue management problem. LP-R and FD-R improve the performances of LP-D and FD-D by incorporating randomization. Interestingly, FD-R appears to be one of the best solution methods, whereas FD-D appears to be one of the worst. Nevertheless, despite its superior performance, FD-R may be less desirable in practice since the right side of (44) has to be computed for each itinerary and the number of itineraries can be large. Similarly, LR-A consistently provides superior performance, but it may be less desirable in practice since the bid-prices used by the decision rule in (43) depend on the remaining leg capacities. Consequently, LR and LP-R seem to emerge as two solution methods that provide a balance between solution quality and practical tractability.

9 Appendix

We show Lemmas 4 and 5 in this section.

Proof of Lemma 4 Associating the dual variables $\{\psi^n_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$ with constraints (14) and (15), after simple algebraic manipulations, the dual of problem (13)-(17) can be written as

$$\max \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} f_j \, \psi^n_{jt} \tag{45}$$

$$\text{subject to} \sum_{j \in \mathcal{J}} p^n_{jt} \, a_{ij} \, \psi^n_{j1} + \ldots + \sum_{j \in \mathcal{J}} p^n_{jt} \, a_{ij} \, \psi^n_{j,t-1} + a_{ij} \, \psi^n_{jt} \leq n \, p^n_{jt} \, c_i \quad \text{for all } i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}^n \tag{46}$$

$$0 \leq \psi^n_{jt} \leq p^n_{jt} \quad \text{for all } j \in \mathcal{J},\ t \in \mathcal{T}^n. \tag{47}$$

Constraints (46) are associated with the decision variables $\{\lambda^n_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}^n\}$, whereas constraints (47) are associated with the decision variables $\{\eta^n_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$ in problem (13)-(17). Some of the complementary slackness conditions for problem (13)-(17) are

$$\left[p^n_{jt} - \hat{\psi}^n_{jt}\right] \hat{\eta}^n_{jt} = 0 \tag{48}$$

$$\Big[\hat{\eta}^n_{jt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1} - f_j\Big] \, \hat{\psi}^n_{jt} = 0 \tag{49}$$

with the boundary condition that $\hat{\rho}^n_{i,n\tau+1} = 0$. Considering constraints (46) for time period $n\tau$, adding these constraints over all $j \in \mathcal{J}$ and noting that $\sum_{j \in \mathcal{J}} p^n_{j,n\tau} = 1$, we obtain

$$\sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} a_{ij} \, \hat{\psi}^n_{jt} \leq n c_i. \tag{50}$$

If we have $f_j > \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}$, then constraints (14) and (15) imply that $\hat{\eta}^n_{jt} > 0$ and we obtain $\hat{\psi}^n_{jt} = p^n_{jt}$ by (48). If we have $f_j < \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1}$, then nonnegativity of $\hat{\eta}^n_{jt}$ implies that $\hat{\eta}^n_{jt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\lambda}^n_{ijt} + \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\rho}^n_{i,t+1} - f_j > 0$ and we obtain $\hat{\psi}^n_{jt} = 0$ by (49). Using these observations in the objective function of problem (45)-(47) shows that (19) holds, whereas using these observations in (50) shows that (20) holds. □

Proof of Lemma 5 We let $\hat{\psi}^1 = \{\hat{\psi}^1_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}^1\}$ be the optimal solution to problem (45)-(47) when solved with $n = 1$. We have

$$V_1^{\hat{\lambda}^1}(c \,|\, 1) = \sum_{t \in \mathcal{T}^1} \sum_{j \in \mathcal{J}} f_j \, \hat{\psi}^1_{jt} = \frac{1}{n} \sum_{t \in \mathcal{T}^n} \sum_{j \in \mathcal{J}} f_j \, \hat{\psi}^1_{j\lceil t/n \rceil},$$

1	2	...	$n-1$	n
$\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j2}$	...	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,n-1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{jn}$
$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,n+1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,n+2}$	...	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,2n-1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,2n}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-2)n+1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-2)n+2}$	...	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-1)n-1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-1)n}$
$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-1)n+1}$	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,(\lceil t/n\rceil-1)n+2}$	...	$+\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,t-1}$	$+a_{ij}\, \tilde\psi^n_{jt}$

Table 3: List of the terms in $\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j1} + \ldots + \sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,t-1} + a_{ij}\, \tilde\psi^n_{jt}$.

1	2	...	$n-1$	n
$\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j1}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j1}$	...	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j1}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j1}$
$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j2}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j2}$	...	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j2}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j2}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j,\lceil t/n\rceil-1}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j,\lceil t/n\rceil-1}$	...	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j,\lceil t/n\rceil-1}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j,\lceil t/n\rceil-1}$
$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$	...	$+\sum_{j\in\mathcal{J}} p^1_{j\lceil t/n\rceil}\, a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$	$+a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$

Table 4: List of the terms in $\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j1} + \ldots + \sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,t-1} + a_{ij}\, \tilde\psi^n_{jt}$ after replacing p^n_{jt} with $p^1_{j\lceil t/n\rceil}$ and $\tilde\psi^n_{jt}$ with $\hat\psi^1_{j\lceil t/n\rceil}$.

where the first equality follows from (18) and the fact that the dual of problem (13)-(17) is problem (45)-(47). If we show that $\{\hat\psi^1_{j\lceil t/n\rceil} : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$ is a feasible solution to problem (45)-(47), then we obtain $V_1^{\hat\lambda^n}(nc\,|\,n) \geq \sum_{t\in\mathcal{T}^n} \sum_{j\in\mathcal{J}} f_j\, \hat\psi^1_{j\lceil t/n\rceil}$ and the result follows. Letting $\tilde\psi^n_{jt} = \hat\psi^1_{j\lceil t/n\rceil}$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}^n$, the remainder of the proof shows that $\tilde\psi^n = \{\tilde\psi^n_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}^n\}$ is a feasible solution to problem (45)-(47).

Table 3 lists the terms in $\sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j1} + \ldots + \sum_{j\in\mathcal{J}} p^n_{jt}\, a_{ij}\, \tilde\psi^n_{j,t-1} + a_{ij}\, \tilde\psi^n_{jt}$. This table has n columns and $\lceil t/n\rceil$ rows. The first $\lceil t/n\rceil - 1$ rows have n entries. The last row has $t - (\lceil t/n\rceil - 1)n$ entries, which is strictly less than n when t is not a multiple of n. Replacing p^n_{jt} with $p^1_{j\lceil t/n\rceil}$ and $\tilde\psi^n_{jt}$ with $\hat\psi^1_{j\lceil t/n\rceil}$ in Table 3, we obtain Table 4. Following the same argument used to obtain (50), we have

$$\sum_{t\in\mathcal{T}^1} \sum_{j\in\mathcal{J}} a_{ij}\, \hat\psi^1_{jt} \leq c_i.$$

Therefore, other than the column that includes the entry $a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$, the sum of the entries in each column in Table 4 is less than or equal to $p^1_{j\lceil t/n\rceil}\, c_i$. Since the solution $\hat\psi^1$ is feasible to problem (45)-(47) when solved with $n = 1$, constraints (46) imply that the sum of the entries in the column that includes the entry $a_{ij}\, \hat\psi^1_{j\lceil t/n\rceil}$ in Table 4 is less than or equal to $p^1_{j\lceil t/n\rceil}\, c_i$. Therefore, the sum of all of the entries in Table 4 is less than or equal to $n\, p^1_{j\lceil t/n\rceil}\, c_i$. The entries in Tables 3 and 4 are identical. Since we have

$p^1_{j\lceil t/n\rceil} = p^n_{jt}$, the sum of all of the entries in Table 3 is less than or equal to $n\, p^n_{jt}\, c_i$ and the solution $\tilde{\psi}^n$ satisfies constraints (46) in problem (45)-(47). Since the solution $\hat{\psi}^1$ is feasible to problem (45)-(47) when solved with $n = 1$, constraints (47) imply that $0 \leq \hat{\psi}^1_{j\lceil t/n\rceil} = \tilde{\psi}^n_{jt} \leq p^1_{j\lceil t/n\rceil} = p^n_{jt}$ and the solution $\tilde{\psi}^n$ satisfies constraints (47) in problem (45)-(47). □

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