ORIE 6334 Bridging Continuous and Discrete Optimization Oct. 30, 2019

Problem Set 3

Due Date: November 13, 2019

As a reminder, the collaboration policy from the syllabus is as follows:

> Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You may not use papers or books or other sources (e.g. material from the web) to help obtain your solution.

1. This problem concerns generalized Laplacians and the Colin de Verdiére invariant as discussed in lecture; we will show that if G is outerplanar and 2-vertex-connected, then $\mu(G) \leq 2$. Recall that an outerplanar graph is one such that there is a planar embedding in which all the vertices of the graph are on the external face. It is known that a graph is outerplanar iff it doesn't contain K_4 or $K_{2,3}$ as a minor. In what follows assume G is outerplanar and 2-vertex connected, and that we have a plane embedding of G in which all the vertices are on the external face F. Since G is 2-vertex-connected and outerplanar, we can assume that F is a cycle.

   (a) Let M be a generalized Laplacian of G with one negative eigenvalue. Assume that $\mu(G) > 2$. Argue that there is $x \in \ker(M)$ such that there are two adjacent vertices u and v with $x(u) = x(v) = 0$.

   (b) Assume that x has minimal support, and that $x \in \ker(M)$ has two adjacent vertices u and v with $x(u) = x(v) = 0$. We will prove that F cannot contain vertices from both $\text{supp}^+(x)$ and $\text{supp}^-(x)$. To do this, suppose otherwise, and pick some $p \in \text{supp}^+(x)$ and $q \in \text{supp}^-(x)$, both in F. Without loss of generality, assume that the vertices are ordered u, v, p, q going in clockwise order around F. Let v' be the first vertex not in $\text{supp}(x)$ going counterclockwise around F from p, and let u' be the first vertex not in $\text{supp}(x)$ going clockwise around F from q. The vertices u', v', p, and q must all be distinct and appear in that order around F. Prove that there must be a path N from v' to q in which every vertex in the path except v' is in $\text{supp}^-(x)$; similarly, there must be a path P from u' to p in which every vertex in the path except u' is in $\text{supp}^+(x)$. Use the existence of these two paths to argue that the graph cannot be planar, and derive a contradiction.

(c) Conclude the proof and show that if G is outerplanar, then it must be the case that $\mu(G) \leq 2$.

2. Let $\mathcal{E}(f)$ be the energy of s-t electrical flow f in a graph G. Let p be the associated potentials. Recall that $\mathcal{E}(f) = p^T L_G p$, and that $p(s) - p(t) = r_{\text{eff}}(s,t)$. Prove that for all vectors $x \in \Re^n$ for which $x(s) - x(t) = r_{\text{eff}}(s,t)$,

$$p^T L_G p \leq x^T L_G x.$$

That is, the potentials p minimize the quadratic form $x^T L_G x$ among all vectors such that the potential difference $x(s) - x(t) = r_{\text{eff}}(s,t)$.

3. (Rayleigh's Monotonicity Principle) Given a graph G, let $\mathcal{E}(f,r)$ be the energy of a flow for supply vector b under resistances r. Let f be the electrical flow for supply vector b under resistances r, and let f' be the electrical flow for the same supply vector b under resistances $r' \geq r$. Prove that $\mathcal{E}(f',r') \geq \mathcal{E}(f,r)$.

4. Let G be an electrical network with resistances r. Prove that effective resistances obey the triangle inequality; that is, for any i, j, k,

$$r_{\text{eff}}(i,k) \leq r_{\text{eff}}(i,j) + r_{\text{eff}}(j,k).$$