ORIE 6334 Bridging Continuous and Discrete Optimization October 2, 2019

Problem Set 2

Due Date: October 23, 2019

As a reminder, the collaboration policy from the syllabus is as follows:

> Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You may not use papers or books or other sources (e.g. material from the web) to help obtain your solution.

1. Let $\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_n(M)$ be the eigenvalues of any matrix $M \in \Re^{n\times n}$. The *Courant-Weyl* inequalities state that for symmetric real matrices A and B,
$$\lambda_i(A+B) \leq \lambda_j(A) + \lambda_{i-j+n}(B)$$
for $1 \leq i \leq j \leq n$ and
$$\lambda_i(A+B) \geq \lambda_j(A) + \lambda_{i-j+1}(B)$$
for $1 \leq j \leq i \leq n$.

   (a) Prove the inequalities. (Hint: recall the proof of the interlacing theorem. Now we need to think about three different vector spaces, for $A+B$, A, and B).

   (b) Use the inequalities to prove a type of interlacing theorem for Laplacians. Consider two graphs on the same vertex set, $G = (V, E)$ and $H = (V, E')$ in which $E' = E - \{e\}$ for a single edge $e \in E$. As usual, we assume that the eigenvalues of Laplacians are ordered as $\lambda_1(L_G) \leq \cdots \leq \lambda_n(L_G)$. Then prove that
$$0 = \lambda_1(L_H) = \lambda_1(L_G) \leq \lambda_2(L_H) \leq \lambda_2(L_G) \leq \cdots \leq \lambda_n(L_H) \leq \lambda_n(L_G).$$
(Aside: one cute application of this theorem is to show that the Petersen graph is not Hamiltonian by showing that these inequalities are violated by the spectrum of the Petersen graph and the spectrum of a cycle on 10 vertices (C_{10}), so that C_{10} is not a subgraph of the Petersen graph.)

2. Prove that the number of spanning trees in K_n, the complete graph on n vertices, is n^{n-2}.

3. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two edge-disjoint graphs on the same vertex set. Let $G = (V, E_1 \cup E_2)$.

   (a) Prove that the algebraic connectivity of the two graphs is superadditive; that is,
   $$\lambda_2(L_{G_1}) + \lambda_2(L_{G_2}) \leq \lambda_2(L_G).$$

   (b) For any graph G, let H be a spanning subgraph of G. Infer that
   $$\lambda_2(L_H) \leq \lambda_2(L_G).$$

4. In this exercise, we'll look at a different way of bounding the largest eigenvalue and obtaining an approximation algorithm for the maximum cut problem. Let λ_n be the maximum eigenvalue of the normalized Laplacian $\mathcal{L}$, and let y be the corresponding eigenvector, with $\max_i |y(i)| \leq 1$. Let OPT denote the number of edges in a maximum cut, and let $S^* \subset V$ denote the set of vertices associated with that set, so that $|\delta(S^*)| = OPT$.

   (a) Prove that if $OPT \geq (1-\epsilon)|E|$, then $\lambda_n \geq 2(1-\epsilon)$.

   (b) Suppose we construct a solution $x \in \{-1, 0, +1\}^n$ as in Trevisan's algorithm (that is, pick $t \in (0, 1]$ uniform, and let $x(i) = -1$ if $y(i) \leq -\sqrt{t}$, $x(i) = 1$ if $y(i) \geq \sqrt{t}$, and $x(i) = 0$ otherwise). Let sets $L = \{i \in V : x(i) = -1\}$, $R = \{i \in V : x(i) = 1\}$, $S = L \cup R$, and $V - S = \{i \in V : x(i) = 0\}$. Prove that for all $0 \leq \beta \leq 1$,
   $$E\left[|\delta(L, R)| + \beta|\delta(S)|\right] \geq \beta(1-\beta) \sum_{(i,j)\in E} (y(i) - y(j))^2.$$

   It might help to know Bergström's inequality, which states that for $a, b \geq 0$ and $0 \leq \beta \leq 1$,
   $$\beta(1-\beta)(a+b)^2 \leq (1-\beta)a^2 + \beta b^2.$$

   (c) Consider $\rho(G) = \max_{S \subset V} \rho(S)$, where
   $$\rho(S) = \max_{\text{partition } S \text{ into } L,R} \frac{|\delta(L, R)| + \frac{1}{2}|\delta(S)|}{|E(S)| + |\delta(S)|}.$$

   Prove that if $\lambda_n \geq 2(1-\epsilon)$, then
   $$E\left[|\delta(L, R)| + \beta|\delta(S)|\right] \geq 2(1-\epsilon)\beta(1-\beta)E\left[2|E(S)| + |\delta(S)|\right].$$

   (d) Set $A = 2(1-\epsilon)\beta(1-\beta)$, and restrict $\frac{1}{2} \leq A + \beta < 1$. Prove that we can use the algorithm to find an S, L, and R such that
   $$\rho(G) \geq \frac{|\delta(L, R)| + \frac{1}{2}|\delta(S)|}{|E(S)| + |\delta(S)|} \geq \frac{1 - 2\beta}{2(1 - A - \beta)}.$$

(e) Use the above to find an α-approximation algorithm for the maximum cut problem for as large an α as you can. Getting $\alpha \geq .529$ will result in full credit. If you can get $\alpha > .614$, you have a publishable paper.

5. (Not a PS problem, no need to answer). The analysis in the problem above is unsatisfying in a couple of ways, at least from a pedagogical standpoint. Unlike the graph parameter $\beta(G)$ presented in class, the parameter $\rho(G)$ more clearly seems to have something to do with finding a large cut in the graph. It would be nice to deal with $\rho(G)$ directly, and prove a Cheeger-style inequality directly on $\rho(G)$, of the form

$$\sqrt{c_1 \lambda_n} \leq \rho(G) \leq c_2 \lambda_n,$$

for some constants c_1, c_2. Perhaps we need to consider a variant $\rho'(G)$ instead, where

$$\rho'(G) = \max_{S \subset V} \max_{\text{partition } S \text{ into } L,R} \frac{|\delta(L,R)| + \frac{1}{2}|\delta(S)|}{\text{vol}(S)},$$

and show that

$$\sqrt{c_1 \lambda_n} \leq \rho'(G) \leq c_2 \lambda_n.$$

It would be even nicer if the analysis of this inequality followed that of Trevisan in some way (e.g. an application of the Cauchy-Schwarz inequality). And finally, it would be very nice if that analysis could lead to an α-approximation algorithm for MAX CUT for some constant $\alpha > .5$. Is any of this possible in a way that could be presented cleanly?