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TECHNICAL NOTE

Contours of Liapunov Functions[1]

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Abstract. As is well known, the stability of a dynamical system in two dimensions may be demonstrated in a very intuitive fashion from the existence of a suitable positive-definite Liapunov function, providing the contours of this function in a neighborhood of the stable point are Jordan curves. It is shown that the Liapunov function will certainly have this property if the stable point is an isolated stationary point in the sense of the Clarke calculus, but a counterexample is given if this assumption is weakened to the stable point being an isolated local extremum.

Key Words. Stability of dynamical systems, Liapunov functions, positive-definite Liapunov functions, level sets, Jordan curves, Clarke derivatives.

1. Introduction

The differential equation

$$\dot{x}(t) = f(x), \tag{1}$$

where the dot denotes derivative with respect to t and f: $W \to \mathbb{R}^n$, $W \subset \mathbb{R}^n$, is used to represent various physical phenomena. For example, $x(t)$ may denote the position and velocity of a particle in $\mathbb{R}^{n/2}$ at a time t, and (1)

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may represent Newton's equation coupled with some constitutive equations. If for some $\bar{x} \in \mathbb{R}^n$, we have $f(\bar{x}) = 0$, and $f(\cdot)$ is Lipschitz at 0, then with the boundary condition $x(0) = \bar{x}$, we have $\dot{x}(t) = 0$ for all t. We call $\bar{x}$ an equilibrium point, since the differential equation implies $x(t) = \bar{x}$ for all t. Physically, this means, if the particle is located at $\bar{x}$, it has always been there, and it will remain there.

The standard technique for testing the stability of equilibrium points is via Liapunov functions. Loosely speaking, an equilibrium point is stable if all nearby solutions stay nearby; for a formal mathematical definition, see Ref. 1. If in addition all nearby points tend toward the equilibrium point as time progresses, the point is said to be asymptotically stable. Almost 100 years ago, Liapunov discovered some conditions which guarantee the stability of an equilibrium point. Liapunov's stability theorem may be stated as follows.

Theorem 1.1. (Liapunov). Let $\bar{x} \in W$ be an equilibrium point of (1). Suppose that, for some neighborhood U of $\bar{x}$, there exists a continuous function $V: U \to \mathbb{R}$ which is continuously differentiable on $U \backslash \{\bar{x}\}$ and satisfies the following conditions:

(i) $V(\bar{x}) = 0$,
(ii) $V(x) > 0$, for all $x \in U \backslash \{\bar{x}\}$,
(iii) $\dot{V} := \nabla V(x) \cdot f(x) \leq 0$, in $U \backslash \{\bar{x}\}$.

Then, $\bar{x}$ is stable. If the last condition is strengthened to

(iv) $\dot{V} < 0$, in $U \backslash \{\bar{x}\}$,

then $\bar{x}$ is asymptotically stable.

A function $V(\cdot)$ which satisfies conditions (i)–(iii) is called a Liapunov function. If condition (iv) is also satisfied, it is a strict Liapunov function. When Eq. (1) is motivated physically, and $\bar{x}$ is suspected to be a stable equilibrium point, usually the energy of the system is a good candidate for a Liapunov function. Condition (iii) implies that no energy is added to the system.

An analytic proof for Liapunov's result may be found in Ref. 1 or Ref. 2. In looking at these proofs, one discovers that the differentiability of $V(\cdot)$ is never used, except in conditions (iii) and (iv). The proof still goes through if we relax our conditions so that $V(\cdot)$ is continuous and replace condition (iii) with

(iii′) $V(x(s)) \leq V(x(t))$, whenever $s > t$, $x(s)$, $x(t) \in U$, and $x(\cdot)$ satisfies (1), and condition (iv) with

(iv′) $V(x(s)) < V(x(t))$, whenever $s > t$, $x(s)$, $x(t) \in U$, and $x(\cdot)$ satisfies (1).

The reader is referred to Ref. 3, p. 89. If a Liapunov function in question is only assumed to be continuous, we shall assume that it satisfies the appropriate primed equation.

The analytic proof, however, gives no intuitive insight into how the Liapunov function gives stability. In the case $n = 2$, Hirsch and Smale (Ref. 1) suggest looking at the contours of $V(\cdot)$,

$$V_d = \{x \in U \mid V(x) = d\},$$

and note that, if V_d is a Jordan curve (simple closed curve) surrounding $\bar{x}$ for all sufficiently small $d > 0$, then condition (3) implies that particles stay inside these level sets. As we shall see, conditions (i), (ii), and (iv) imply that the curves V_d are indeed Jordan curves nested around $\bar{x}$, and it is this fact which gives the intuitive basis to a geometric proof. Some authors (Refs. 4–5) present their stability proofs informally by asserting that this contour property is automatically true from conditions (i) and (ii). Some care is needed here: consider the function

$$V(x, y) = \begin{cases} y^2 + x^2 + 4x^3 \sin(1/x), & x \neq 0, \\ y^2, & x = 0. \end{cases}$$

This function is continuously differentiable and has a strict local minimum at the origin. It can be verified that this function has a sequence of strict local extrema approaching the origin along the positive x-axis, so that there does not exist an $\epsilon > 0$ such that the contours V_d are Jordan curves for all $d \in (0, \epsilon)$.

This motivates us to study the geometry of positive-definite functions and their level sets. It is interesting to compare our results with the techniques used in the classical approach. For example, Theorem 3.1 in Chapter 1 of Ref. 6 shows that, if V satisfies conditions (1) and (2) in our Theorem 1.1, then the local level sets are closed surfaces with respect to $\bar{x}$. That is to say, any continuous path from $\bar{x}$ to the boundary of U must pass through each local level set. By contrast, our concern is in identifying conditions in the two-dimensional case ensuring that these level sets conform to our geometric intuition that they should be Jordan curves surrounding $\bar{x}$. This is of course not necessary for the approach adopted in Ref. 6, but we believe that our results help toward a better understanding of the method of Liapunov functions and in diverse other areas concerned with local minima of functions (cf. Ref. 7).

There has been a huge amount of recent interest in generalizations of dynamical systems and control theory to such areas as differential inclusions and nonsmooth control (see Refs. 7–8 for examples). It therefore is natural to ask the above geometric questions in cases where Liapunov functions are not known to be smooth (see Chapter 6 of Ref. 8 for examples).

For the majority of this study, we shall divorce our attention from the connection to the differential equation (1). We will require some results from planar topology, so we shall make references to Ref. 9.

2. Main Results

The first result presented gives a very general fact about the connectedness of level sets for a positive-definite function. A related result is found in Ref. 10. We denote the open ball in $\mathbb{R}^2$ with center $\bar{x}$ and radius r as $B(\bar{x}; r)$.

Theorem 2.1. Suppose that $V: U \to \mathbb{R}$ is continuous, $U \subset \mathbb{R}^2$ is open, and $\bar{x} \in U$. Suppose also that $V(\bar{x}) = 0$ and $V(x) > 0$ for $x \in U\backslash\{\bar{x}\}$. Then, if $V(\cdot)$ has no local minima in $U\backslash\{\bar{x}\}$, there exists $r > 0$ and $\delta > 0$ such that, for all $d \in (0, \delta)$, the lower level set

$$\underline{V}_d = \{x \in B(\bar{x}; r) \mid V(x) < d\}$$

is connected. If in addition $V(\cdot)$ contains no local maxima in $U\backslash\{\bar{x}\}$, then $\underline{V}_d$ is simply connected, the contour

$$V_d = \{x \in B(\bar{x}; r) \mid V(x) = d\}$$

is connected and is exactly the boundary of $\underline{V}_d$, $\mathrm{cl}(\underline{V}_d)/\underline{V}_d$, and the upper level set

$$\bar{V}_d = \{x \in B(\bar{x}; r) \mid V(x) > d\}$$

is connected.

Proof. Since $x \in U$, and U is open, we can choose $r > 0$ so that $B(\bar{x}; r) \subset U$. Define δ by

$$\delta = \min\{V(x) \mid \|x - \bar{x}\| = r\}.$$

By compactness, $\delta > 0$. Choose any $d \in (0, \delta)$ and suppose $\underline{V}_d$ is not connected. Then

$$\underline{V}_d = S_1 \cup S_2,$$

where S_1 and S_2 are open, disjoint, and nonempty. Suppose without loss of generality that $\bar{x} \in S_1$. Since $\mathrm{cl}(S_2)$ is compact, there exists a $y \in \mathrm{cl}(S_2)$ which minimizes $V(\cdot)$ on $\mathrm{cl}(S_2)$, hence $V(y) < d$. If $y \in \mathrm{cl}(S_2)\backslash S_2$, this would imply by continuity that $V(y) \geq d$, which is a contradiction, so $y \in S_2$. Then, y is a local minimum of $V(\cdot)$, since S_2 is open, and $y \neq x$, contradicting our hypothesis. Thus, $\underline{V}_d$ is connected.

If $V(\cdot)$ also has no local maxima on $U\backslash\{\bar{x}\}$, suppose that $\underline{V}_d$ is not simply connected. Then, for some Jordan curve J contained in $\underline{V}_d$, which bounds a nonempty, open, simply connected set S_3, there is some $z \in S_3$ such that $V(z) \geq d$ (see Ref. 11, p. 104). This implies that the maximum of $V(\cdot)$ on $S_3 \cup J$, denoted by $\bar{z}$, satisfies $V(\bar{z}) \geq d$. Since $\bar{z} \notin J$, $V(\cdot)$ attains a local maximum at $\bar{z}$, a contradiction, so $\underline{V}_d$ is simply connected. If $u \in V_d$, but $u \notin \mathrm{cl}(\underline{V}_d)$, then in some neighborhood of u, $V(x) \geq d$, so u is a local minimum, a contradiction. Thus, $V_d \subset \mathrm{cl}(\underline{V}_d)\backslash V_d$, and the reverse inclusion holds by continuity. Finally, the fact that V_d and $\bar{V}_d$ are connected follows from Ref. 9, pp. 143–144. □

This gives us a lot of information regarding the contours and leads to a natural question: if S is a bounded, open, simply connected set, is $\mathrm{cl}(S)\backslash S$ a Jordan curve? Unfortunately, as is well known, it may not be. Consider the following set:

$$S = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1,\ -3 < y < f(x)\},$$

where

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ -1, & x = 0. \end{cases}$$

This set is clearly bounded, open, and simply connected, but its boundary $\mathrm{cl}(S)\backslash S$ is not a Jordan curve. Pictorially, the problem lies with the points on the y-axis, with $y \in (-1, 1]$. These points belong to $\mathrm{cl}(S)\backslash S$, but they are not accessible from S. A point p is accessible from a set S if there exists a Jordan arc with one endpoint at p lying entirely in $\{p\} \cup S$. A property of Jordan curves is that each point p on the curve is accessible from both its interior and exterior. Schönflies used this result to obtain a converse of the Jordan curve theorem (see Ref. 9).

Theorem 2.2. (Schönflies). Suppose that the sets $A, G_1, G_2 \subset \mathbb{R}^2$ satisfy the following properties:

(i) A is compact,
(ii) G_1 and G_2 are both open, nonempty, and connected,
(iii) $\mathbb{R}^2\backslash A = G_1 \cup G_2$,
(iv) $G_1 \cap G_2 = \varnothing$,
(v) each $p \in A$ is accessible from both G_1 and G_2.

Then, A is a Jordan curve.

Clearly, we could replace $\mathbb{R}^2$ with any open ball B which contains A. Using this theorem, we can give conditions which imply that the contours near the minimum of a Liapunov function are Jordan curves. Notice that, since the Schönflies theorem does not extend simply to higher dimensions, this fairly simple approach will not go through immediately in higher dimensions. For example, every point on the torus in three dimensions is clearly accessible from its interior and exterior, yet it is not homeomorphic to the sphere. Despite this, generalizations of the following result to $n > 2$, $n \neq 5$, are possible using much more advanced techniques (see Refs. 12–13). For $n = 5$, the question is in fact equivalent to the Poincaré conjecture (see Ref. 12).

Theorem 2.3. Suppose that, for some open neighborhood $U \subset \mathbb{R}^2$ of $\bar{x}$, there exists a continuous function $V: U \to \mathbb{R}$ which is continuously differentiable on $U \backslash \{\bar{x}\}$ and satisfies the following conditions:

(i) $V(\bar{x}) = 0$,
(ii) $V(x) > 0$, for all $x \in U \backslash \{\bar{x}\}$,
(iii) for all $x \in U \backslash \{\bar{x}\}$, $\nabla V(x) \neq 0$.

Then, there exists some neighborhood $W \subset U$ of $\bar{x}$ and an $\epsilon > 0$ such that the contours

$$V_d = \{x \in W \mid V(x) = d\}$$

are Jordan curves for each $d \in (0, \epsilon)$, each of which surrounds $\bar{x}$.

Proof. The conditions imply that Theorem 2.1 is applicable. We choose $W = B(\bar{x}, r)$ as in Theorem 2.1, and then identifying A with V_d, G_1 with $\underline{V}_d$, and G_2 with $\bar{V}_d$, all the conditions of the Schönflies theorem are satisfied, with the exception of accessibility. To show that any $p \in V_d$ is accessible from both $\underline{V}_d$ and $\bar{V}_d$, we just need to construct ascent and descent directions at any $x \neq \bar{x}$. The technique is standard: consider the arc, parametrized by t,

$$x(t) = p + t\nabla V(p).$$

Using Taylor's theorem about $t = 0$, we obtain

$$\begin{aligned} V(x(t)) &= V(p + t\nabla V(p)) \\ &= d + t\nabla V(p) \cdot \nabla V(p + \bar{t}\nabla V(p)), \end{aligned}$$

where $\bar{t} \in (0, t)$. As $t \to 0$, $\bar{t} \to 0$, hence condition (iii) and the continuity of ∇V imply that there is some $t^* > 0$ such that $\{x(t) \mid t \in [0, t^*]\}$ is an arc giving the accessibility of p from $\bar{V}_d$, and $\{x(t) \mid t \in [-t^*, 0]\}$ is an arc giving the accessibility of p from $\underline{V}_d$. We then apply the Schönflies theorem to deduce that V_d is a Jordan curve. This holds for each $d \in (0, \epsilon)$. □

If $V(\cdot)$ is a strict Liapunov function for an equilibrium point of Eq. (1), the conditions for Theorem 2.3 are satisfied, so its contours actually match intuition. We can weaken this condition by noting that, if the Liapunov function is only continuous [i.e., we replace condition (iv) with (iv′)], the points of V_d are still accessible from $\bar{V}_d$ and $\underline{V}_d$ by considering the arc $x(\cdot)$ from Eq. (1).

Upon consideriing what we have done so far, we see that, though Theorem 2.1 requires $V(\cdot)$ to be only continuous, Theorem 2.3 assumes $V(\cdot)$ to be C^1 to obtain accessibility. To weaken the amount of regularity required to achieve accessibility, we would have to modify condition (iii) of Theorem 2.3. We do this by appealing to the Clarke calculus.

In generalizing calculus to nondifferentiable functions, Clarke (Refs. 7 and 14) introduced the notion of generalized gradients and directional derivatives. For the class of locally Lipschitz functions, the theory extends effectively. This generalized derivative of a function $V(\cdot)$, which we denote by $\partial V(\cdot)$, is a set-valued function. When the function is continuously differentiable, this set contains one element which is the derivative. If the function is convex, the set matches the subderivative which is obtained in convex analysis (see Ref. 15).

Using this more general theory, we may now extend our results.

Lemma 2.1. Suppose $V(\cdot)$ is a locally Lipschitz function and $V(x) = \alpha$. Then, if $0 \notin \partial V(x)$, x is accessible from $\underline{V}_\alpha$ and $\bar{V}_\alpha$.

Proof. Again, we simply need to construct ascent and descent directions at x; cf. Ref. 7, Section 6.2. Since $0 \notin \partial V(x)$, there exists d such that the generalized directional derivative $V^0(x, d) < 0$; see Ref. 7, Theorem 2.1.5. Then, we must have $x + \lambda d \in \underline{V}_\alpha$ for all λ sufficiently small. If not, then there exists a positive sequence $\lambda_m \downarrow 0$, with $V(x + \lambda_m d) \geq \alpha$, so that for all m,

$$[V(x + \lambda_m d) - V(x)]/\lambda_m \geq 0.$$

This leads to the contradiction

$$0 > V^0(x, d) = \lim_{\delta \downarrow 0} \sup_{\substack{0 < t < \delta \\ \|y - x\| < \delta}} [V(y + td) - V(y)]/t \geq 0.$$

We thus have accessibility from $\underline{V}_\alpha$. Accessibility from $\bar{V}_\alpha$ is proved similarly. □

The proof above relies only on the construction of a descent direction. This is possible using a variety of generalized derivatives in place of Clarke's; see for example Ref. 16 for a survey.

If $0 \in \partial V(x)$, we say that x is a stationary point. The following result gives a substantial generalization of Theorem 2.3.

Theorem 2.4. Suppose that $U \subset \mathbb{R}^2$ is open and $V: U \to \mathbb{R}$ is locally Lipschitz with a strict local extremum $\bar{x}$ which is an isolated stationary point. Then, the contours of $V(\cdot)$ in a neighborhood of $\bar{x}$ are Jordan curves surrounding $\bar{x}$.

Proof. The proof is the same as that of Theorem 2.3, except for the accessibility part, which is proved by Lemma 2.1. □

A somewhat different, more topological approach to this problem appeals to a deformation-invariant definition of a critical point; see Ref. 17. Suppose that $V: U \to \mathbb{R}$ is a continuous function, where $U \subset \mathbb{R}^n$ is open. A point $\bar{x} \in U$ is an essentially regular point of $V(\cdot)$ if there exist open neighborhoods D and $\bar{D}$ of $\bar{x}$ in U and an $\bar{x}$-preserving homeomorphism $\phi: \bar{D} \to D$ such that $k = V \circ \phi$ is a nonconstant affine function on $\bar{D}$. All other points of U are called essentially critical points.

Notice that any local extremum is an essentially critical point. On the other hand, the origin is not an essentially critical point of the function $y = x^3$, although it is a critical point.

Theorem 2.5. Suppose that $U \subset \mathbb{R}^2$ is open and $V: U \to \mathbb{R}$ is continuous with a strict local extremum $\bar{x}$ which is an isolated essentially critical point. Then, the contours of $V(\cdot)$ in a neighborhood of $\bar{x}$ are Jordan curves surrounding $\bar{x}$.

Proof. As observed above, $\bar{x}$ is an isolated local extremum, so Theorem 2.1 applies. It remains to show that, if x_0 is an essentially regular point, then x_0 is accessible from $\underline{V}_d$ and $\bar{V}_d$, where $d = V(x_0)$. By essential regularity, there are neighborhoods $\bar{D}$ and D of x_0 and a homeomorphism $\phi: \bar{D} \to D$ preserving x_0 so that, for any $x \in \bar{D}$,

$$V(\phi(x)) - V(\phi(x_0)) = a \cdot (x - x_0),$$

for some nonzero $a \in \mathbb{R}^2$. Thus, if we define $x(t) := x_0 + ta$, then $x(t) \in \bar{D}$ for all small t, so $\phi(x(t)) \in D$ and $V(\phi(x(t))) = d + t\|a\|^2$. So for some $t^* > 0$, $\{\phi(x(t)) \mid t \in [0, t^*]\}$ is a Jordan arc giving accessibility from x_0 to $\bar{V}_d$, and similarly for $\underline{V}_d$. □

In an attempt to weaken the conditions further, we now outline the construction of a function F which satisfies all the conditions of Theorem 2.4 or Theorem 2.5, with the exception that $\bar{x}$ is simply an isolated local extremum, rather than the stronger condition of being an isolated stationary point or isolated essentially critical point. This function will not have contours near $\bar{x}$ which are all Jordan curves.

3. Counterexample

Define an even function $g: [-1, 1] \to \mathbb{R}$ by $g(2^{-n}) = (-1)^n + 2$, for $n \in \mathbb{Z}^+$, g linear on each interval $[2^{-n}, 2^{1-n}]$, and $g(0) = 1$. Then, define a set $T \subset \mathbb{R}^2$ as the union of the graph of g with the line segments $[(0, 1), (0, 3)]$, $[(1, 3), (1, -1)]$, $[(1, -1), (-1, -1)]$, and $[(-1, -1), (-1, 3)]$. Let T_1 be the set bounded by T,

$$T_1 := \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < g(x)\};$$

define the distance function $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \mathrm{dist}((x, y); T)$, and define $F: \mathbb{R}^2 \to \mathbb{R}$ initially by

$$F(x, y) = \begin{cases} 1 - f, & \text{on } T_1, \\ 1, & \text{on } T, \\ 1 + f, & \text{otherwise.} \end{cases}$$

The function $F(\cdot)$ clearly has a unique global extremum (minimum) at $(0, 0)$ with value 0. It also has a global Lipschitz constant of one. To show that $(0, 0)$ is the only local extrema, one may interpret the function through the radius of the largest ball which just touches T. For any point $(x, y) \notin \{(0, 0) \cup T\}$, it is not difficult to see that the ball corresponding to such a point may be expanded or contracted continuously by moving the center in a particular direction. This also gives accessibility, which implies that F_d is a Jordan curve for $d \in (-1, 0) \cup (0, \infty)$. Since the contour $F_0 = T$, this is clearly not a Jordan curve.

Upon examining this function more closely, it is also not hard to see that, for d sufficiently close to -1 or d sufficiently large, the contours of F corresponding to these values are star-shaped about $(0, 0)$. The contours consist of line segments and arcs of circles of constant radius. Even stronger, we can actually parametrize any one of these contours in polar coordinates by a Lipschitz function of an angular variable θ about the center $(0, 0)$. Let $g_1(\cdot)$ be the function corresponding to such a contour Γ_1 close to $(0, 0)$, and let $g_2(\cdot)$ be the function corresponding to such a contour Γ_2 outside T.

We are now going to redefine F inside Γ_1, so cut out the region enclosed by Γ_1. Make a copy of the region enclosed by Γ_2, and scale it by a factor $\alpha < 1$ with origin $(0, 0)$, so that it fits strictly into the cut-out region. The boundary of this shrunken region (Γ_3, say) is parametrized by $\alpha g_2(\cdot)$; see Fig. 1.

Call v_1 the value of F on the contour Γ_1; call v_2 the value of F on Γ_2; and let us redefine $F \equiv v_3$ on Γ_3, where $v_3 < v_1$. Redefine F on the annular region between Γ_3 and Γ_1 using polar coordinates by

$$F(r, \theta) = v_3 + (v_1 - v_3)\left[\frac{r - \alpha g_2(\theta)}{g_1(\theta) - \alpha g_2(\theta)}\right].$$

This function is Lipschitz, since g_1 and g_2 are Lipschitz; and providing v_3 is sufficiently close to v_1, the Lipschitz constant 1 will be maintained. Notice that F is strictly increasing in r for constant θ.

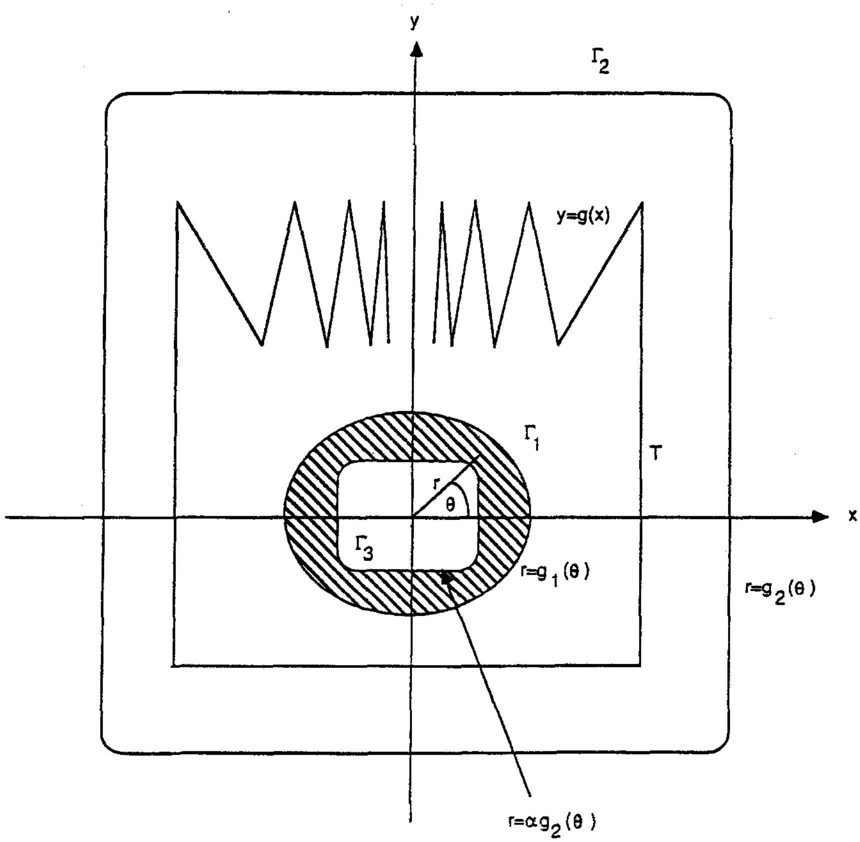

Fig. 1. Level sets of counterexample.

We have that F is Lipschitz 1 between and on $\alpha\Gamma_2$ and Γ_2, with $F \equiv v_2$ on Γ_2 and $F \equiv v_3$ on $\alpha\Gamma_2$, where $v_3 < v_2$ and $0 < \alpha < 1$. Suppose that F has been defined between and on $\alpha^{p+1}\Gamma_2$ and $\alpha^p\Gamma_2$, $p = 0, 1, \ldots, m$. Define F between and on $\alpha^{m+2}\Gamma_2$ and $\alpha^{m+1}\Gamma_2$ by

$$F(x, y) := \alpha F((1/\alpha)(x, y)) + v_3 - \alpha v_2.$$

Allowing m to increase, we can define F on the entire region enclosed by Γ_2 except the origin. We define

$$F(0, 0) = (v_3 - \alpha v_2)/(1 - \alpha).$$

To see this, first note that we may inductively prove that

$$F \equiv \frac{1}{1-\alpha}(v_3 - \alpha v_2 + (v_2 - v_3)\alpha^m), \qquad \text{on } \alpha^m\Gamma_2.$$

Secondly, F is well defined, since the previous result shows that

$$F(\alpha^{m+1}\Gamma_2) = \alpha F(\alpha^m\Gamma_2) + v_3 - \alpha v_2, \qquad m = 0, 1, \ldots,$$

as required. Lastly, F is Lipschitz 1 on the region enclosed by Γ_2. If $F(\cdot)$ is Lipschitz 1 on and between $\alpha^{m+1}\Gamma_2$ and $\alpha^m\Gamma_2$, then $\alpha F(1/\alpha)$ is Lipschitz 1 on and between $\alpha^{m+2}\Gamma_2$ and $\alpha^{m+1}\Gamma_2$, so induction shows F is Lipschitz 1 on the region without the origin. Since F is defined by continuity at the origin, the result follows.

Now each region between $\alpha^{m+1}\Gamma_2$ and $\alpha^m\Gamma_2$ has a scaled, shifted copy of F between Γ_2 and $\alpha\Gamma_2$. Thus, we do not introduce any local extrema, $(0, 0)$ remains a strict minimizer, and yet each of the regions (which shrink to the origin) contains a contour of F which is not a Jordan curve.

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