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VARIATIONAL ANALYSIS OF PSEUDOSPECTRA*

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Abstract. The ϵ-pseudospectrum of a square matrix A is the set of eigenvalues attainable when A is perturbed by matrices of spectral norm not greater than ϵ. The pseudospectral abscissa is the largest real part of such an eigenvalue, and the pseudospectral radius is the largest absolute value of such an eigenvalue. We find conditions for the pseudospectrum to be Lipschitz continuous in the set-valued sense and hence find conditions for the pseudospectral abscissa and the pseudospectral radius to be Lipschitz continuous in the single-valued sense. Our approach illustrates diverse techniques of variational analysis. The points at which the pseudospectrum is not Lipschitz (or more properly, does not have the Aubin property) are exactly the critical points of the resolvent norm, which in turn are related to the coalescence points of pseudospectral components.

Key words. pseudospectrum, variational analysis, Lipschitz multifunction, Aubin property, nonsmooth analysis, normal cone

AMS subject classifications. 15A18, 49K40, 65F15, 65K10, 90C31

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1. Introduction.

Analysis using eigenvalues is prevalent in many different areas of applied mathematics. As we consider perturbations to an $n \times n$ complex matrix A with spectrum $\Lambda(A)$, we are led to study the ϵ-pseudospectrum $\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$, which is a set-valued map defined by

$$\Lambda_\epsilon(A) = \{z \mid \exists E \in M^n \text{ such that } \|E\| \leq \epsilon, z \in \Lambda(A+E)\},$$

where M^n is the space of matrices of size $n \times n$. A well-known equivalent formulation, assuming $\|\cdot\| = \|\cdot\|_2$ as we do throughout, is

$$\Lambda_\epsilon(A) = \{z \mid \underline{\sigma}(A - zI) \leq \epsilon\},$$

where $\underline{\sigma}(A)$ denotes the smallest singular value of the matrix A. As discussed extensively in [22], the function $z \mapsto (zI - A)^{-1}$ is called the *resolvent* of the matrix A. Thus the pseudospectra of A are just upper-level sets of the *resolvent norm* function $n_A : \mathbb{C} \backslash \Lambda(A) \to \mathbb{R}_+$ defined by

$$n_A(z) := \left\|(zI - A)^{-1}\right\| = \frac{1}{\underline{\sigma}(A - zI)}.$$

Pseudospectra may be more informative than eigenvalues in applications where matrices are nonnormal [22, 13].

The continuity of the spectrum is well known [14]. One immediate question is whether continuity extends to Λ_ϵ. Since Λ_ϵ is a set-valued map, we ask whether we have continuity in the Hausdorff metric, and it is known that the answer is yes [17, Theorem 2.3.7].

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Name of Property	Mathematical Statement
$\boxed{\text{Lipschitz Continuity}}$	
Definition $\updownarrow$	
$\boxed{\text{Aubin Property}}$	Λ_ϵ Aubin at A for z
Mordukhovich Criterion $\updownarrow$	
$\boxed{\text{Coderivatives}}$	$D^*\Lambda_\epsilon\,(A \mid z)\,(0) = \{0\}$
Definition of Coderivatives $\updownarrow$	
$\boxed{\text{Normals of gph}\Lambda_\epsilon}$	$(M^n \times \{0\}) \cap N_{\operatorname{gph}\Lambda_\epsilon}(A, z) = \{0\}$
Level sets $\updownarrow$	
$\boxed{\text{Subgradients of } \underline{\sigma}^e}$	$(M^n \times \{0\}) \cap \mathbb{R}_+ \partial \underline{\sigma}^e\,(A, z) = \{0\}$
Toeplitz–Hausdorff theorem $\updownarrow$	
$\boxed{\text{Numerical Range}}$	$0 \notin Y\,(A - zI)$
Subdifferential Calculus $\updownarrow$	
$\boxed{\text{Singular Values}}$	$0 \notin \partial\,(-\underline{\sigma}_A)\,(z)$

FIG. 1. *Equivalences of properties summarized in Theorem* 5.2.

Does the pseudospectrum mapping Λ_ϵ have stronger continuity properties? One of the main contributions of this paper is to find conditions under which the map Λ_ϵ is Lipschitz continuous. The ingredients of our analysis are variational-analytic techniques from the last couple of decades, as described in Rockafellar and Wets [21], Clarke et al. [10], and Mordukhovich [20]. In particular, we should note that there are technical details involved in the generalization of Lipschitz continuity to set-valued maps. Our proof (of the main results in Theorem 5.2 and Proposition 6.3) may be described loosely by Figure 1. The reader may find the schematic outline helpful as the argument proceeds.

For the moment, we remark on the notation

$$\underline{\sigma}_A\,(z) = \underline{\sigma}^e\,(A, z) = \frac{1}{n_A\,(z)}$$

and $Y\,(A - zI)$, which refers to the set of the inner products of associated left and right singular vectors (see page 1050). N refers to the normal cone, ∂ refers to the subdifferential and D^* refers to the coderivative. We expand more on the notation of Figure 1 (see page 1051).

In Figure 1, the six properties on the right on A and z are equivalent. For a given matrix A, we call points z not satisfying these equivalent properties "resolvent-critical" because they are smooth or nonsmooth critical points of the norm of the resolvent n_A. When the multiplicity of the smallest singular value of $A - zI$ is one, this property is equivalent to z being a "degenerate point" (in the sense of [4, Definition 4.5, Corrigendum]) or not "regular" in the sense of [5, Definition 4.4]. Points not resolvent-critical are exceptional for several aspects of pseudospectra, notably the quadratic convergence of the pseudospectral abscissa algorithm in [5].

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As well as our main result equating the absence of the Aubin property with resolvent-criticality, we derive a variety of other properties of resolvent-critical points proving, in particular, that points where pseudospectral components coalesce as ϵ grows are resolvent-critical.

As an application of the Lipschitz continuity of $\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$, we find conditions for the Lipschitz continuity (in the single-valued sense) and the strict differentiability of the pseudospectral abscissa $\alpha_\epsilon : M^n \to \mathbb{R}$, and the pseudospectral radius $\rho_\epsilon : M^n \to \mathbb{R}_+$ defined by

$$\alpha_\epsilon (A) := \max \{\mathrm{Re}(\lambda) \mid \lambda \in \Lambda_\epsilon (A)\},$$
$$\rho_\epsilon (A) := \max \{|\lambda| \mid \lambda \in \Lambda_\epsilon (A)\}.$$

We write $MSV : M^n \rightrightarrows \mathbb{C}^n \times \mathbb{C}^n$, with

$$MSV(A) := \{(u,v) \mid u, v \text{ minimal left and right singular vectors of } A\}.$$

In the above definition of MSV, u, v are *minimal left and right singular vectors* of A if they are unit vectors satisfying

$$\underline{\sigma}(A)\, u = Av$$
$$\text{and } \underline{\sigma}(A)\, v = A^H u,$$

where A^H is the Hermitian transpose of A. A key tool in our analysis is the set

$$Y(A) := \left\{ v^H u \mid (u,v) \in MSV(A) \right\}.$$

We prove that the set $Y(A - zI)$ is the subgradient set at z of the function $-\underline{\sigma}_A : \mathbb{C} \to \mathbb{R}_-$, where $\underline{\sigma}_A(z) = \underline{\sigma}(A - zI)$.

Related to Λ_ϵ is the mapping $\Lambda_\epsilon^c : M^n \rightrightarrows \mathbb{C}$ defined by $\Lambda_\epsilon^c(A) = \{z \mid \underline{\sigma}(A - zI) \geq \epsilon\}$. This mapping turns out to be easier to analyze because $-\underline{\sigma}(\cdot)$ has the property of subdifferential regularity (as defined in [21]) except at where it is zero. We show that the normal cone $N_{\Lambda_\epsilon^c(A)}(\bar{z})$ is $\mathbb{R}_+ (Y(A - \bar{z}I))$. This establishes a link between the variational properties of $-\underline{\sigma}_A$ and Λ_ϵ^c, and the Aubin property.

Notation. For future reference, Tables 1 and 2 summarize the mappings that appear throughout the paper.

Unless otherwise stated, our notation follows [21]. See also the table of notation in [21, page 725]. The term “regular” means subdifferentially regular in the sense of [21, Definition 7.25]. Table 2 summarizes the symbols we use.

The “^{H}” in A^H and v^H represent the Hermitian transpose of a matrix or vector, while the “*” in L^* represents the adjoint of the linear operator L. Note that D^* stands for the coderivative instead. The real inner product on $A, B \in M^n$ is defined by $\mathrm{Re}\ \mathrm{tr}\ \left(A^H B\right)$.

Outline. The paper is organized as follows. Section 2 studies the continuity properties of the pseudospectra Λ_ϵ and its “complement” Λ_ϵ^c via more general feasible-set mappings. In sections 3, 4, and 5, we prove the main result that Λ_ϵ has the Aubin property at A for z if and only if $0 \notin Y(A - zI)$, with section 3 containing general results on variational analysis and the singular value decomposition, section 4 performing subdifferential calculus, and section 5 finishing the proof of the main result.

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Table 1
Summary of definitions.

Name/domain/range	Definition
$\bar{\sigma} : M^n \to \mathbb{R}_+$	$\bar{\sigma}(A)$ is maximum singular value of A
$\underline{\sigma} : M^n \to \mathbb{R}_+$	$\underline{\sigma}(A)$ is minimum singular value of A
$\underline{\sigma}^e : M^n \times \mathbb{C} \to \mathbb{R}_+$	$\underline{\sigma}^e(A, z) = \underline{\sigma}(A - zI)$
$\underline{\sigma}_A : \mathbb{C} \to \mathbb{R}_+$	$\underline{\sigma}_A(z) = \underline{\sigma}(A - zI)$
$\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$	$\Lambda_\epsilon(A) = \{z \mid \underline{\sigma}(A - zI) \le \epsilon\}$
$\Lambda : M^n \rightrightarrows \mathbb{C}$	$\Lambda(A) = \Lambda_0(A) = \{\text{eigenvalues of } A\}$
$\Lambda_\epsilon^c : M^n \rightrightarrows \mathbb{C}$	$\Lambda_\epsilon^c(A) = \{z \mid \underline{\sigma}(A - zI) \ge \epsilon\}$
$\alpha_\epsilon : M^n \to \mathbb{R}$	$\alpha_\epsilon(A) = \max_{z \in \Lambda_\epsilon(A)} \operatorname{Re} z$
$\rho_\epsilon : M^n \to \mathbb{R}_+$	$\rho_\epsilon(A) = \max_{z \in \Lambda_\epsilon(A)} \lvert z \rvert$
$W : M^n \rightrightarrows \mathbb{C}$	Numerical range/ field of values[15, Definition 1.1.1]
$MSV : M^n \rightrightarrows \mathbb{C}^n \times \mathbb{C}^n$	See Definition 3.2
$Y : M^n \rightrightarrows \mathbb{C}$	See Definition 3.2

Table 2
Summary of definitions.

Symbol	Explanation	Reference from [21]
$\hat{\partial}$	regular subgradient set	Definition 8.3
∂	subgradient set	Definition 8.3
∂^∞	horizon subgradient set	Definition 8.3
$\hat{N}$	regular normal cone	Definition 6.3
N	normal cone	Definition 6.4
osc	outer semicontinuous	Definition 5.4
isc	inner semicontinuous	Definition 5.4
pos	positive hull	section 3G
$\operatorname{lip} S(\cdot \mid \cdot)$	graphical modulus	Definition 9.36
$\operatorname{lip}_\infty S(\cdot)$	Lipschitz modulus	Definition 9.28
limsup	(set) outer limit	Formula 5(1)
liminf	(set) inner limit	Formula 5(1)
$D^* S(\cdot \mid \cdot)$	coderivative	Definition 8.33
$\lvert\cdot\rvert^+$	outer norm	Formula 9(4)
$\mathbf{d}(\cdot, \cdot)$	Pompieu-Hausdorff distance	Example 4.13
$\operatorname{lev}_{\le \alpha} f$	Level sets: $\{x \mid f(x) \le \alpha\}$	section 1B
conv	convex hull	section 1E
bdry	boundary of a set	
$\mathbb{B}$	unit ball $\{x \mid \lvert x \rvert \le 1\}$	

In section 6, we show how the Lipschitz constant for the map Λ_ϵ can be calculated. Section 7 gives conditions for the Lipschitz continuity and strict differentiability of the pseudospectral abscissa α_ϵ and the pseudospectral radius ρ_ϵ. Finally, we present properties of resolvent-critical points in section 8. We prove, in particular, that the points at which the components of $\Lambda_\epsilon(A)$ coalesce as ϵ grows are resolvent-critical, and we pose some questions about resolvent-critical points.

2. Feasible-set mappings and continuity of pseudospectra. The pseudospectral mapping $\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$ has two inputs: $\epsilon \in \mathbb{R}_+$ and the matrix in the argument of $\Lambda_\epsilon(\cdot)$. As $\mathbb{R}_+$ is one-dimensional, the variation of $\Lambda_\epsilon(A)$ for a fixed matrix A and variable ϵ is easier to visualize, as implemented in EigTool [24]. Some attractive results in this direction have been obtained in [7, 8, 18, 1, 17] and elsewhere. By contrast, in this work we study how Λ_ϵ behaves for a fixed ϵ and a varying matrix argument, primarily taking a more abstract and systematic approach than [6].

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We study pseudospectra using the language of set-valued analysis as described in the monograph [21]. We take the definition of inner semicontinuity and outer semicontinuity in [21, section 5B].

In the next two propositions, let $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m$ be a continuous function, and let $T : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be the mapping defined by

$$T(w) = \{x \mid f(x, w) \in D\}, \tag{2.1}$$

where D is a closed set.

PROPOSITION 2.1. *T is outer semicontinuous.*

Proof. We just need to check that T has a closed graph (by [21, Theorem 5.7]), which is easy. ∎

Note that the ϵ-pseudospectrum can be written as

$$\begin{aligned} \Lambda_\epsilon(A) &= \{z \mid \underline{\sigma}^e(A, z) \le \epsilon\} \\ &= \{z \mid \underline{\sigma}^e(A, z) \in (-\infty, \epsilon]\}. \end{aligned}$$

If we apply Proposition 2.1, we can deduce that Λ_ϵ is outer semicontinuous. In a similar manner, Λ_ϵ^c, defined by $\Lambda_\epsilon^c(A) = \{z \mid \underline{\sigma}^e(A, z) \ge \epsilon\}$, is also outer semicontinuous.

Turning to inner semicontinuity, we begin with a technical result.

PROPOSITION 2.2. *Let*

$$Q := \operatorname{cl}\{x \mid f(x, \bar{w}) \in \operatorname{int}(D)\},$$

so $Q \subset T(\bar{w})$. We have

(a) *$Q \subset \liminf_{w \to \bar{w}} T(w) \subset T(\bar{w})$.*

In the case where $m = 1$:

(b) *If $D = (-\infty, \alpha]$, then*

$$\begin{aligned} Q = &\{x \mid f(x, \bar{w}) = \alpha \text{ , } x \text{ is not a local minimizer of } f(\cdot, \bar{w})\} \\ &\cup \{x \mid f(x, \bar{w}) < \alpha\}. \end{aligned}$$

(c) *If $D = [\alpha, \infty)$, then*

$$\begin{aligned} Q = &\{x \mid f(x, \bar{w}) = \alpha \text{ , } x \text{ is not a local maximizer of } f(\cdot, \bar{w})\} \\ &\cup \{x \mid f(x, \bar{w}) > \alpha\}. \end{aligned}$$

(d) *If $\alpha > 0$, f is positively homogeneous (that is, $\lambda f(\cdot) = f(\lambda \cdot)$ for $\lambda > 0$) and either $D = (-\infty, \alpha]$ or $D = [\alpha, \infty)$, then $Q = \liminf_{w \to \bar{w}} T(w)$.*

Proof. Property (a) is easy and standard. See, for example, the techniques in [2, 16].

Statements (b) and (c) are clear by the definition of Q, so we proceed to prove statement (d) for the case $D = (-\infty, \alpha]$. (The case $D = [\alpha, \infty)$ is similar and is omitted.) From statement (a), we just need to prove that if $\bar{x} \notin Q$, then $\bar{x} \notin \liminf_{w \to \bar{w}} T(w)$. Suppose that $\bar{x} \notin Q$. We need to consider only $\bar{x} \in T(\bar{w})$, so we can assume that $\bar{x}$ is a minimizer of $f(\cdot, \bar{w})$ and $f(\bar{x}, \bar{w}) = \alpha$. Then there is a neighborhood $\mathbb{B}_\delta(\bar{x})$ about $\bar{x}$ such that $f(x, \bar{w}) \ge f(\bar{x}, \bar{w}) = \alpha$ if $x \in \mathbb{B}_\delta(\bar{x})$. If $\|x - \bar{x}\| < \delta/2$, then

$$\left\| \frac{1}{1 + \frac{1}{j}} x - \bar{x} \right\| < \delta \text{ if } j \text{ is large.}$$

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This means that

$$\begin{aligned}
f\left(x,\left(1+\frac{1}{j}\right)\bar{w}\right) &= \left(1+\frac{1}{j}\right) f\left(\frac{1}{1+\frac{1}{j}}x,\bar{w}\right) \\
&\geq \left(1+\frac{1}{j}\right)\alpha \quad \left(\text{because } \left\|\left(\frac{1}{1+\frac{1}{j}}\right)x-\bar{x}\right\| < \delta\right) \\
&> \alpha,
\end{aligned}$$

which implies that $\mathbb{B}_{\delta/2}(\bar{x})\cap T((1+\frac{1}{j})\bar{w}) = \emptyset$ if j is large enough. So for the sequence $(1+\frac{1}{j})\bar{w} \to \bar{w}$ as $j\to\infty$, we cannot find a subsequence x_j such that $x_j \in T((1+\frac{1}{j})\bar{w})$ and $x_j \to \bar{x}$, and this means that $\bar{x} \notin \liminf_{w\to\bar{w}} T(w)$. ∎

The following corollary is immediate from the definition of inner semicontinuity.

COROLLARY 2.3. *If $T(\bar{w}) = Q$, then T is continuous at $\bar{w}$. Furthermore, if f is positively homogeneous, then the converse holds as well.*

Proof. The mapping T is continuous if and only if it is both inner and outer semicontinuous. Apply the last two propositions. ∎

Now that we have established conditions for outer and inner semicontinuity for feasible-set mappings, we shall study the continuity of the pseudospectrum Λ_ϵ and Λ_ϵ^c. Let us consider the case $\epsilon = 0$ first. The map $\Lambda_0^c : M^n \rightrightarrows \mathbb{C}$ is not interesting as $\Lambda_0^c(A) = \mathbb{C}$ for all matrices A. We are then led to consider the spectrum $\Lambda_0 = \Lambda$, which is well known to be continuous [14, Appendix D].

To extend to $\epsilon > 0$, we may apply Propositions 2.1 and 2.2, combined with the fact that $\underline{\sigma}_A(\cdot)$ has no local minimum other than at the eigenvalues [22, Theorem 2.4(i)], to prove the following result. This result is not new and can be found, for example, in [17, Corollary 2.3.8].

PROPOSITION 2.4. *$\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$ is continuous for $\epsilon \geq 0$.*

For $\Lambda_\epsilon^c : M^n \rightrightarrows \mathbb{C}$, we obtain the following using Proposition 2.2(d).

PROPOSITION 2.5. *$\Lambda_\epsilon^c : M^n \rightrightarrows \mathbb{C}$ is outer semicontinuous, but it is inner semicontinuous at a matrix A if and only if there is no local maximizer $\bar{z}$ to $\underline{\sigma}_A : \mathbb{C} \to \mathbb{R}_+$, with $\underline{\sigma}_A(\bar{z}) = \epsilon$.*

Example 2.6. The mapping Λ_ϵ^c is not continuous at some points. For a concrete example of the noncontinuity of Λ_ϵ^c, consider the point $0 \in \Lambda_1^c(\bar{A})$, where $\bar{A} = \operatorname{diag}(1,-1,i,-i)$ and $\epsilon = 1$. Here $\Lambda_1(\bar{A})$ consists of the union of balls of radius 1 around the diagonal entries, and so we observe that 0 is a local maximum of $\underline{\sigma}_{\bar{A}}$. This exhibits an example of the discontinuity of Λ_1^c as $\liminf_{A\to\bar{A}} \Lambda_1^c(A) \subsetneq \Lambda_1^c(\bar{A})$.

Next, we consider Lipschitz continuity. First, we define the Pompieu–Hausdorff distance.

DEFINITION 2.7 (see [21, Example 4.13]). *For $C, D \subset \mathbb{R}^n$ closed and nonempty, the* Pompieu–Hausdorff distance $\mathbf{d}(C,D)$ *is defined as*

$$\mathbf{d}(C,D) := \inf\{\eta \geq 0 \mid C \subset D + \eta\mathbb{B}, D \subset C + \eta\mathbb{B}\}.$$

Lipschitz continuity is thus defined as follows.

DEFINITION 2.8 (see [21, Definitions 9.26, 9.28]). *A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is* Lipschitz continuous *if it is nonempty–closed-valued and there exists $\kappa \in \mathbb{R}_+$, a Lipschitz constant, such that $\mathbf{d}(S(x), S(x')) \leq \kappa |x - x'|$ for all $x, x' \in \mathbb{R}^n$, or*

$$S(x') \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in \mathbb{R}^n.$$

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The infimum of all κ such that there exists a neighborhood V of $\bar{x}$ such that

$$S(x') \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in V$$

is the Lipschitz modulus *for S at $\bar{x}$ and is denoted by* $\mathrm{lip}_\infty S(\bar{x})$.

The Aubin property, which is a localized Lipschitz property, is defined as follows.

DEFINITION 2.9 (see [21, Definition 9.36] Aubin property and graphical modulus). *A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the* Aubin property at $\bar{x}$ for $\bar{u}$, *where $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in S(\bar{x})$, if* $\mathrm{gph}\, S$ *is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods V of $\bar{x}$ and W of $\bar{u}$, and a constant $\kappa \in \mathbb{R}_+$ such that*

$$S(x') \cap W \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in V.$$

The graphical modulus *of S at $\bar{x}$ for $\bar{u}$, denoted by* $\mathrm{lip}\, S(\bar{x} \mid \bar{u})$, *is the infimum of all such κ that satisfy the formula above.*

If the function f in the feasible-set mapping in formula (2.1) in page 1052 is smooth, we understand the Aubin Property quite well through [21, Example 9.51]. If $D = (-\infty, \bar{\alpha}]$, we can also analyze the nonsmooth case. In what follows, ∂ and ∂^∞ denote, respectively, the subgradient set and the horizon subgradient set [21, Definition 8.3].

Assumptions (a), (b), and (c) in the result below are standard for computing normals to level sets (see, for example, [21, Proposition 10.3].) Assumption (d) is needed to apply a chain rule.

THEOREM 2.10. *Consider the set-valued map $C : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ defined via a level set representation*

$$C(p) = \{x \mid F(x, p) \le \bar{\alpha}\},$$

with $F : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$. Suppose that

(a) $F(\bar{x}, \bar{p}) = \bar{\alpha}$,

(b) $(0, 0) \notin \partial F(\bar{x}, \bar{p})$,

(c) *F is regular at $(\bar{x}, \bar{p})$,*

(d) $(0, y_2) \in \partial^\infty F(\bar{x}, \bar{p}) \implies y_2 = 0$.

Then C has the Aubin property at $\bar{p}$ for $\bar{x}$ if and only if $0 \notin \partial F_{\bar{p}}(\bar{x})$, where $F_{\bar{p}} : \mathbb{R}^n \to \mathbb{R}$ is defined by $F_{\bar{p}}(x) := F(x, \bar{p})$. In this case,

$$\mathrm{lip}\; C(\bar{p} \mid \bar{x}) = \max_{\substack{(a,b) \in N_{\mathrm{gph}\, C}(\bar{p}, \bar{x}) \\ \|b\| = 1}} \|a\|,$$

If $F(\bar{x}, \bar{p}) < \bar{\alpha}$, then C has the Aubin property at $\bar{p}$ for $\bar{x}$, with $\mathrm{lip}\; C(\bar{p} \mid \bar{x}) = 0$.

Proof. The Mordukhovich criterion [21, Theorem 9.40] tells us that C has the Aubin property at $\bar{p}$ for $\bar{x}$ if and only if $D^* C(\bar{p} \mid \bar{x})(0) = \{0\}$, where D^* denotes the coderivative [21, Definition 8.33]. This holds if and only if

$$(z, 0) \in N_{\mathrm{gph}\, C}(\bar{p}, \bar{x}) \text{ implies } z = 0. \tag{2.2}$$

This property is equivalent to

$$(0, z) \in N_{\mathrm{gph}\, C^{-1}}(\bar{x}, \bar{p}) \text{ implies } z = 0.$$

Conditions (a), (b), and (c) allow us to conclude that

$$N_{\mathrm{gph}\, C^{-1}}(\bar{x}, \bar{p}) = (\mathrm{pos}\, \partial F(\bar{x}, \bar{p})) \cup \partial^\infty F(\bar{x}, \bar{p}) \tag{2.3}$$

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through a result on level sets [21, Proposition 10.3], or

$$(0,z) \in (\text{pos}\, \partial F(\bar{x},\bar{p})) \cup \partial^\infty F(\bar{x},\bar{p}) \text{ implies } z = 0,$$

and by condition (d), this is in turn equivalent to

$$(0,z) \in \text{pos}\, \partial F(\bar{x},\bar{p}) \text{ implies } z = 0. \tag{2.4}$$

We define $L_{\bar{p}} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$ by $L_{\bar{p}}(x) = (x,\bar{p})$. The adjoint $L_{\bar{p}}^* : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is given by $L_{\bar{p}}^*(x,p) = x$. We have $F_{\bar{p}} = F \circ L_{\bar{p}}$, and so by a chain rule [21, Theorem 10.6] and condition (d), $\partial F_{\bar{p}}(\bar{x}) = L_{\bar{p}}^* \partial F(\bar{x},\bar{p})$. Thus

$$\begin{aligned} \partial F_{\bar{p}}(\bar{x}) &= L_{\bar{p}}^* \partial F(\bar{x},\bar{p}) \\ &= \{y \mid \exists z \text{ such that } (y,z) \in \partial F(\bar{x},\bar{p})\}. \end{aligned}$$

If $0 \in \partial F_{\bar{p}}(\bar{x})$, then there exists z such that $(0,z) \in \partial F(\bar{x},\bar{p})$, but condition (b) implies $z \neq 0$, which contradicts statement (2.4). If $0 \notin \partial F_{\bar{p}}(\bar{x})$, this means that there is no z such that $(0,z) \in \partial F(\bar{x},\bar{p})$ and implies statement (2.4). So $0 \notin \partial F_{\bar{p}}(\bar{x})$ is equivalent to C not having the Aubin property at $\bar{p}$ for $\bar{x}$ as claimed.

The calculation of the value $\text{lip}\, C(\bar{p} \mid \bar{x})$ follows from the definition of the coderivative $D^*C(\bar{p} \mid \bar{x})$ and its relation to the normal cone through the Mordukhovich criterion. If $F(\bar{x},\bar{p}) < \bar{\alpha}$, then the normal cone is $\{(0,0)\}$, giving us the required value of $\text{lip}\, C(\bar{p} \mid \bar{x})$. ∎

To obtain the Lipschitz modulus from the graphical modulus, one may use [21, Theorem 9.38], but Proposition 6.2 is sufficient for our purposes in this paper.

In sections 3 to 6, we will be using the general principle illustrated in Theorem 2.10 to study where the pseudospectrum Λ_ϵ has the Aubin property and also to illustrate how this can identify where Λ_ϵ is Lipschitz continuous and give a value of the Lipschitz constant.

One may immediately try to apply Theorem 2.10 to show that Λ_ϵ has the Aubin property for A at z. In this case, $p = A$, $x = z$, and so $C(p) = \Lambda_\epsilon(A)$, $F(x,p) = \underline{\sigma}(A - zI) = \underline{\sigma}^e(A,z)$. However, $\underline{\sigma}^e$ is not a regular function, but this can be overcome by studying $-\underline{\sigma}^e$ instead, which is regular if $A - zI$ is nonsingular. This is what we will do in the analysis that follows.

3. General results.

First, we are interested in finding out whether the functions $-\underline{\sigma}^e$ and $\frac{1}{\underline{\sigma}^e}$ enjoy similar regularity properties so that we can deduce properties of $\underline{\sigma}^e$. We recall a result on the reciprocals of functions.

PROPOSITION 3.1 (see [20, Corollary 1.111(iii)]). *For any function $h : \mathbb{R}^n \to \mathbb{R}$ at z where $h(z) > 0$, we have $\partial h(z) = h(z)^2 \partial\left(-\frac{1}{h}\right)(z)$, and h is regular at z if and only if $-\frac{1}{h}$ is regular there.*

The set of minimal singular vectors of A, $MSV(A)$, is defined below.

DEFINITION 3.2. *For a matrix A, the left and right singular vectors corresponding to the smallest singular value of A are the pairs $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$, $\|u\| = \|v\| = 1$, which appear in the appropriate columns of U and V in some singular value decomposition $A = USV^H$ of A. We refer to u and v as* minimal singular vectors*, and we denote the set of pairs of minimal singular vectors of A as $MSV(A)$. Furthermore, we define $Y : M^n \rightrightarrows \mathbb{C}$ by*

$$Y(A) := \left\{v^H u \mid (u,v) \in MSV(A)\right\}.$$

An equivalent definition given in the introduction is to have pairs of unit vectors (u,v) satisfying the equations $\underline{\sigma}(A)u = Av$ and $\underline{\sigma}(A)v = A^H u$.

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The following result summarizes a complete characterization of left and right minimal singular vectors when we have one particular singular value decomposition, which is helpful for the case where the smallest singular value is multiple.

PROPOSITION 3.3. *Consider a matrix $A \in M^n$ with singular value decomposition (for unit vectors u_j, v_j)*

$$A = \sum_{j=1}^{n} \sigma_j u_j v_j^H = USV^H,$$

where $\sigma_1 = \sigma_2 = \cdots = \sigma_m < \sigma_j$ for all $j > m$. Define matrices $\bar{U} = (u_1\, u_2\, \cdots\, u_m)$ and $\bar{V} = (v_1\, v_2\, \cdots\, v_m)$. Then

$$MSV\,(A) = \left\{ \left(\bar{U}q, \bar{V}q \right) \mid q \in \mathbb{C}^m, \|q\| = 1 \right\}$$

if A is invertible (in other words, $\sigma_1 > 0$) and

$$MSV\,(A) = \left\{ \left(\bar{U}q_1, \bar{V}q_2 \right) \mid q_1, q_2 \in \mathbb{C}^m, \|q_1\| = \|q_2\| = 1 \right\}$$

if A is singular.

Proof. The equations $Av = \underline{\sigma}\,(A)\,u$ and $A^H u = \underline{\sigma}\,(A)\,v$ require u to be an eigenvector for AA^H and v to be an eigenvector for $A^H A$, and so they lie in the subspaces spanned by the columns of $\bar{U}$ and $\bar{V}$, respectively. We have assumed that these columns are placed at the left of U and V. Then let $v = \bar{V}q$. As we want a v of unit length, we must have $\|q\| = 1$. Since A is invertible, $\underline{\sigma} := \underline{\sigma}\,(A) > 0$, and so

$$u = \frac{1}{\underline{\sigma}} Av = \frac{1}{\underline{\sigma}} USV^H \bar{V} q = \frac{1}{\underline{\sigma}} US \begin{pmatrix} I \\ 0 \end{pmatrix} q = U \begin{pmatrix} I \\ 0 \end{pmatrix} q = U \begin{pmatrix} q \\ 0 \end{pmatrix} = \bar{U} q.$$

Thus $MSV\,(A) \subset \left\{ \left(\bar{U}q, \bar{V}q \right) \mid q \in \mathbb{C}^m, \|q\| = 1 \right\}$. The reverse is straightforward.

If A is singular, then as before, $u = \bar{U}q_1$ and $v = \bar{V}q_2$ for some unit vectors q_1, q_2. It is evident that u and v satisfy the relations $\underline{\sigma}\,(A)\,u = Av$ and $\underline{\sigma}\,(A)\,v = A^H u$, so we are done. □

The significance of $Y\,(A)$ will become clear later in sections 4 and 5. We first show a result on $Y\,(A)$.

PROPOSITION 3.4. *If A is invertible, then $Y\,(A)$ is convex.*

Proof. We make the observation that the set $Y\,(A)$ can be determined as follows. Let $\bar{U}$ and $\bar{V}$ be as described in Proposition 3.3. The numerical range of a matrix $B \in M^n$ is the set $\left\{ v^H B v \mid v \in \mathbb{C}^n, \|v\| = 1 \right\}$, denoted by $W\,(B)$, and is convex by the Toeplitz–Hausdorff theorem [15, Property 1.2.2]. Then

$$\begin{aligned} Y\,(A) &= \left\{ v^H u \mid (u, v) \in MSV\,(A) \right\} \\ &= \left\{ q^H \bar{V}^H \bar{U} q \mid \|q\| = 1 \right\} \text{ (by Proposition 3.3)} \\ &= W\left(\bar{V}^H \bar{U} \right), \text{ the numerical range of } \bar{V}^H \bar{U}, \end{aligned}$$

establishing the convexity of $Y\,(A)$. □

For singular matrices A, $Y\,(A)$ need not be convex. Take, for example, the singular value decomposition

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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With this matrix,

$$Y(A) = \left\{ q_1 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} q_2 \mid q_1, q_2 \in \mathbb{C}, |q_1| = |q_2| = 1 \right\}$$
$$= \{q \in \mathbb{C} \mid |q| = 1\},$$

which is not convex.

4. Subdifferential calculus. This section collects some results about subdifferential calculus involving $\underline{\sigma}^e : M^n \times \mathbb{C} \to \mathbb{R}_+$, where $\underline{\sigma}^e(A, z) = \underline{\sigma}(A - zI)$. As suggested in Figure 1, there is a link between the subdifferential $\partial\underline{\sigma}^e(A, z)$ and normal cone $N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z)$ for $\underline{\sigma}^e(A, z) = \epsilon$. Before we can apply the appropriate theorems in [21], we have to calculate $\partial\underline{\sigma}^e(A, z)$, establish regularity properties, and characterize whether $0 \in \partial\underline{\sigma}^e(A, z)$.

When the smallest singular value is simple, $\underline{\sigma}$ and $\underline{\sigma}^e$ are analytic, as the next lemmas assert.

We remind the reader that the spaces M^n and $M^n \times \mathbb{C}$ have (real) inner products defined by

$$\langle A, B\rangle = \mathrm{Re}\ \mathrm{tr}\left(A^H B\right) \text{ for } A, B \in M^n$$

and

$$\langle (X, x), (Y, y)\rangle = \mathrm{Re}\left(\mathrm{tr}\left(X^H Y\right) + x^H y\right) \text{ for } X, Y \in M^n \text{ and } x, y \in \mathbb{C}.$$

LEMMA 4.1. *If the invertible matrix A has a simple smallest singular value, then the function $\underline{\sigma} : M^n \to \mathbb{R}_+$ is real-analytic at A, with gradient*

$$\nabla\underline{\sigma}(A) = uv^H$$

for any $(u, v) \in MSV(A)$.

The proof for the above lemma is standard (for example, [4, Theorem 7.1]), while the lemma below follows by noticing that $\underline{\sigma}^e = \underline{\sigma} \circ L$ and applying the chain rule, where $L : M^n \times \mathbb{C} \to \mathbb{C}$ is defined by $L(A, z) = A - zI$.

LEMMA 4.2. *If $z \notin \Lambda(A)$ and $A - zI$ has a simple smallest singular value, then the function $\underline{\sigma}^e : M^n \times \mathbb{C} \to \mathbb{R}_+$ is real-analytic at (A, z), with gradient*

$$\nabla\underline{\sigma}^e(A, z) = \left(uv^H, -v^H u\right)$$

for any $(u, v) \in MSV(A - zI)$.

The next two results are generalizations of Lemmas 4.1 and 4.2 to the nonsmooth case, and they calculate the subgradients needed in the main result in section 5.

PROPOSITION 4.3. *Suppose $z \notin \Lambda(A)$. Then*

$$\partial\left(-\underline{\sigma}^e\right)(A, z) = conv\left\{\left(-uv^H, v^H u\right) \mid (u, v) \in MSV(A - zI)\right\}.$$

Furthermore, $-\underline{\sigma}^e$ is regular at (A, z) and globally Lipschitz.

Proof. We consider the functions

$$\bar{\sigma}^e : M^n \times \mathbb{C} \to \mathbb{R}_+,\ \iota : M^n \to M^n \text{ and } L : M^n \times \mathbb{C} \to M^n$$

defined by

$$\bar{\sigma}^e(A, z) = \bar{\sigma}\left((A - zI)^{-1}\right),\ \iota(B) = B^{-1} \text{ and } L(A, z) = A - zI.$$

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That is, $\bar{\sigma}^e = \bar{\sigma} \circ \iota \circ L$. To evaluate the subdifferential of this function, we apply a chain rule [21, Theorem 10.6]. Given a matrix B, we seek to evaluate $\nabla (\iota \circ L)(A,z)^* (B)$, which is, by the chain rule, $\nabla L(A,z)^*(\nabla \iota(A-zI)^*(B))$.

As $\bar{\sigma}$ is everywhere Lipschitz, $\partial^\infty \bar{\sigma}(\iota \circ L(A,z)) = \{0\}$. Furthermore, since $\bar{\sigma}$ is convex, it is regular at $\iota \circ L(A,z)$, and so the conditions for [21, Theorem 10.6] are satisfied.

It is easy to check the identity $L^*(B) = (B, -\mathrm{tr} B)$. (Note that L is linear so $\nabla L = L$ and $\nabla L^* = L^*$.) Using the binomial expansion

$$(M+\Delta)^{-1} = M^{-1} - M^{-1}\Delta M^{-1} + o(\Delta),$$

it follows that $\nabla \iota(M)(B) = -M^{-1}BM^{-1}$, so $\nabla \iota(M)^*(B) = -M^{-H}BM^{-H}$ follows easily.

Next, we evaluate $\partial \bar{\sigma}^e(A,z)$. Let the singular value decomposition of $(A-zI)$ be USV^H. Then the singular value decomposition of $(A-zI)^{-1}$ is $VS^{-1}U^H$, and $(A-zI)^{-H} = US^{-1}V^H$. So

$$\partial \bar{\sigma}^e(A,z) = \nabla L(A,z)^* \, \nabla \iota (A-zI)^* \, \partial \bar{\sigma}\left((A-zI)^{-1}\right).$$

We know that

$$\partial \bar{\sigma}(B) = \mathrm{conv}\left\{uv^H \mid \|u\| = \|v\| = 1, Bv = \bar{\sigma}(B)u, B^H u = \bar{\sigma}(B)v\right\}.$$

(See, for example, [23].) Therefore,

$$\partial \bar{\sigma}\left((A-zI)^{-1}\right) = \mathrm{conv}\left\{vu^H \mid (u,v) \in MSV(A-zI)\right\}.$$

Then for any $(u,v) \in MSV(A-zI)$, we have

$$\begin{aligned}
\nabla L(A,z)^* \, \nabla \iota(A-zI)^* \left(vu^H\right) &= \nabla L(A,z)^* \left(-US^{-1}V^H vu^H US^{-1}V^H\right) \\
&= \underline{\sigma}(A-zI)^{-2} \, \nabla L(A,z)^* \left(-uv^H\right) \\
&= \underline{\sigma}(A-zI)^{-2} \left(-uv^H, \mathrm{tr}\ \left(uv^H\right)\right) \\
&= \underline{\sigma}(A-zI)^{-2} \left(-uv^H, v^H u\right),
\end{aligned}$$

and so

$$\partial \bar{\sigma}^e(A,z) = \underline{\sigma}(A-zI)^{-2} \, \mathrm{conv}\left\{\left(-uv^H, v^H u\right) \mid (u,v) \in MSV(A-zI)\right\}.$$

By Proposition 3.1, we conclude that

$$\begin{aligned}
\partial\left(-\underline{\sigma}^e\right)(A,z) &= \partial\left(-\frac{1}{\bar{\sigma}^e}\right)(A,z) \\
&= \bar{\sigma}^e(A,z)^{-2} \, \partial \bar{\sigma}^e(A,z) \\
&= \mathrm{conv}\left\{\left(-uv^H, v^H u\right) \mid (u,v) \in MSV(A-zI)\right\}.
\end{aligned}$$

The function $-\underline{\sigma}^e$ is regular at (A,z) because $\bar{\sigma}$ is regular and both the chain rule [21, Theorem 10.6] and Proposition 3.1 guarantee the preservation of regularity. Also, the function $-\underline{\sigma}^e$ is globally Lipschitz because $-\underline{\sigma}^e = -\underline{\sigma} \circ L$ is the composition of two globally Lipschitz functions. ∎

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From the definition of $\Lambda_\epsilon(A) = \{z \mid \underline{\sigma}_A(z) \leq \epsilon\}$, where $\underline{\sigma}_A : \mathbb{C} \to \mathbb{R}_+$ is defined by $\underline{\sigma}_A(z) = \underline{\sigma}(A - zI)$, it is clear that the functions $\underline{\sigma}$ and $\underline{\sigma}_A$ figure prominently in the study of pseudospectra. The following two results can be seen as nonsmooth analogues of [4, Theorem 7.1 and Corollary 7.2]. Even though $\underline{\sigma}$ and $\underline{\sigma}_A$ are not necessarily smooth, we are able to prove that $-\underline{\sigma}$ and $-\underline{\sigma}_A$ are regular and calculate their subgradients.

PROPOSITION 4.4. *The function $-\underline{\sigma}$ is regular at every nonsingular matrix $A \in M^n$, with*

$$\partial(-\underline{\sigma})(A) = -\mathrm{conv}\left\{uv^H \mid (u,v) \in MSV(A)\right\}.$$

Proof. Define $L_{M^n} : M^n \to M^n \times \mathbb{C}$ by $L_{M^n}(A) = (A, 0)$, so we have $-\underline{\sigma}_A = (-\underline{\sigma}^e) \circ L_{M^n}$. Clearly L_{M^n} is smooth, with $\nabla L_{M^n} = I \times \mathbf{0}$ at all points. $(\nabla L_{M^n})^* : M^n \times \mathbb{C} \to M^n$ is just the natural projection. Thus, by appealing to [21, Theorem 10.6] and Proposition 4.3, we get what we need. ∎

PROPOSITION 4.5. *For a matrix A, consider the function $\underline{\sigma}_A : \mathbb{C} \to \mathbb{R}_+$ defined by $\underline{\sigma}_A(z) = \underline{\sigma}(A - zI)$. If $z \notin \Lambda(A)$, then*

$$\partial(-\underline{\sigma}_A)(z) = Y(A - zI),$$

and $-\underline{\sigma}_A$ is regular at z and globally Lipschitz.

Proof. The proof is similar to the proof above, but we work through the details for completeness. We note $-\underline{\sigma}_A = (-\underline{\sigma}^e) \circ L_A$, where $L_A : \mathbb{C} \to M^n \times \mathbb{C}$, $L_A(z) = (A, z)$. Clearly L_A is smooth, with $\nabla L_A = \mathbf{0} \times I$ at all points. Furthermore, $(\nabla L_A)^* : M^n \times \mathbb{C} \to \mathbb{C}$ is just the natural projection. Thus, by appealing to a chain rule [21, Theorem 10.6] and Proposition 4.3, we have

$$\begin{aligned}\partial(-\underline{\sigma}_A)(z) &= (\nabla L_A)^* \partial(-\underline{\sigma}^e)(A, z) \\ &= Y(A - zI).\end{aligned}$$

As in Proposition 4.3, $\underline{\sigma}_A$ is globally Lipschitz because it is a composition of two globally Lipschitz functions. ∎

We note that the assumptions that $A - zI$ is nonsingular in Proposition 4.3 and A is nonsingular in Proposition 4.4 cannot be dropped in the proposition below.

PROPOSITION 4.6. *If $z \in \Lambda(A)$, then $-\underline{\sigma}^e$ is not regular at (A, z). Similarly, $-\underline{\sigma}$ is not regular at A if A is singular.*

Proof. Take $\bar{U}$ and $\bar{V}$ to the matrices corresponding to the minimal left and right singular vectors of $A - zI$ in the statement of Proposition 3.3. For small $\epsilon > 0$, we have

$$\begin{aligned}-\underline{\sigma}^e\left(A + \epsilon\bar{U}\bar{V}^H, z\right) &= -\underline{\sigma}^e(A, z) - \epsilon \\ \text{and } -\underline{\sigma}^e\left(A - \epsilon\bar{U}\bar{V}^H, z\right) &= -\underline{\sigma}^e(A, z) - \epsilon.\end{aligned}$$

Hence if $(B, x) \in \hat{\partial}(-\underline{\sigma}^e)(A, z)$, we have

$$\begin{aligned}-\underline{\sigma}^e\left(A \pm \epsilon\bar{U}\bar{V}^H, z\right) &\geq -\underline{\sigma}^e(A, z) + \left\langle (B, x), \left(\pm\epsilon\bar{U}\bar{V}^H, 0\right)\right\rangle + o(\epsilon) \\ \implies -\epsilon &\geq \epsilon\left\langle (B, x), \left(\pm\bar{U}\bar{V}^H, 0\right)\right\rangle + o(\epsilon).\end{aligned}$$

Dividing by ϵ throughout and taking limits as $\epsilon \downarrow 0$, we have

$$\begin{aligned}&-1 \geq \left\langle (B, x), \left(\pm\bar{U}\bar{V}^H, 0\right)\right\rangle \\ \implies &-2 \geq \left\langle (B, x), \left(\bar{U}\bar{V}^H, 0\right)\right\rangle + \left\langle (B, x), \left(-\bar{U}\bar{V}^H, 0\right)\right\rangle = 0,\end{aligned}$$

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which is obviously a contradiction. This means that $\hat{\partial}\left(-\underline{\sigma}^e\right)(A,z)=\emptyset$. To show that $\partial\left(-\underline{\sigma}^e\right)(A,z)\neq\emptyset$, we note that for small $\epsilon>0$, we have

$$\left(-u_1v_1^H, v_1^H u_1\right)\in\hat{\partial}\left(-\underline{\sigma}^e\right)\left(A+\epsilon\bar{U}\bar{V}^H,z\right)$$

by Proposition 4.3, where the minimal left and right singular vectors u_1, v_1 are defined in the statement of Proposition 3.3. Taking $\epsilon\downarrow 0$, this ensures that $(-u_1v_1^H, v_1^H u_1)\in\partial(-\underline{\sigma}^e)(A,z)$, and thus $\partial\left(-\underline{\sigma}^e\right)(A,z)\neq\emptyset$. Since $\partial\left(-\underline{\sigma}^e\right)$ and $\hat{\partial}\left(-\underline{\sigma}^e\right)$ differ and appealing to [21, Corollary 8.11], $-\underline{\sigma}^e$ is not regular at (A,z). The proof for $-\underline{\sigma}$ is similar. □

PROPOSITION 4.7. *The resolvent norm $n_A:\mathbb{C}\to\mathbb{R}$ defined by $n_A(z)=\|(zI-A)^{-1}\|$ is regular at every point where $z\notin\Lambda(A)$, with*

$$\partial n_A(z)=n_A(z)^2\,Y(A-zI).$$

Proof. From the identity $n_A=1/\underline{\sigma}_A$ and Propositions 3.1 and 4.5, we note the following calculations:

$$\begin{aligned}\partial n_A(z) &= n_A(z)^2\,\partial\left(-\frac{1}{n_A}\right)(z)\\ &= n_A(z)^2\,\partial\left(-\underline{\sigma}_A\right)(z)\\ &= n_A(z)^2\,Y(A-zI).\end{aligned}\qquad\square$$

This motivates the following definition.

DEFINITION 4.8. *A point $z\in\mathbb{C}$ is* resolvent-critical *for a square matrix A if either $z\in\Lambda(A)$ or $0\in Y(A-zI)$.*

Thus resolvent-critical points that are not eigenvalues are simply critical points of the resolvent norm n_A (in the nonsmooth sense). Recall that, for a locally Lipschitz function f, $\partial^\circ f(x)$, the convex hull of $\partial f(x)$, is the *Clarke subdifferential* of f at x and that $\bar{x}$ is *Clarke-critical* if $0\in\partial^\circ f(\bar{x})$. Since $\underline{\sigma}_A$ is globally Lipschitz, the following holds as well.

THEOREM 4.9. *For a given matrix A, the following are equivalent:*

(1) *z is resolvent-critical.*

(2) *z is Clarke-critical for $-\underline{\sigma}_A$.*

(3) *z is Clarke-critical for $\underline{\sigma}_A$.*

Proof. Since $\underline{\sigma}_A$ is Lipschitz, we have $\partial^\circ\left(-\underline{\sigma}_A\right)(z)=-\partial^\circ\underline{\sigma}_A(z)$ by [9, Proposition 2.3.1]. This means that (2) and (3) are equivalent.

Next we prove that (1) implies (2). If z is resolvent-critical, then either z is an eigenvalue of A or $0\in\partial\left(-\underline{\sigma}_A\right)(z)$. In the second case, z is Clarke-critical for $-\underline{\sigma}_A$ because $\partial\left(-\underline{\sigma}_A\right)(z)\subset\partial^\circ\left(-\underline{\sigma}_A\right)(z)$. In the first case, z is a maximizer of $-\underline{\sigma}_A$, and so z is Clarke-critical.

Lastly, we prove that (2) implies (1). If z is not resolvent-critical, then z is not an eigenvalue, and $0\notin\partial\left(-\underline{\sigma}_A\right)(z)$. But $\partial\left(-\underline{\sigma}_A\right)(z)=\partial^\circ\left(-\underline{\sigma}_A\right)(z)$ by the regularity of $-\underline{\sigma}_A$ at z, so we are done. □

Example 4.10. Table 3 shows some examples where 0 is a resolvent-critical point of A. (In the third example, the resolvent-critical point is close to 0 but not exactly at 0.) These plots were obtained with EigTool [24]. The curves represent the boundaries of the pseudospectra $\Lambda_\epsilon(A)$ for $\epsilon=10^\alpha$, where α is the number corresponding to the line generated by EigTool in the legend on the right. The third example is found in [12].

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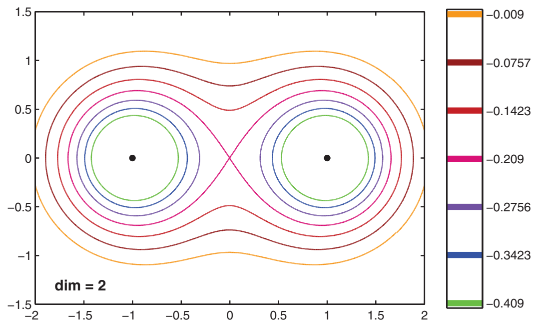
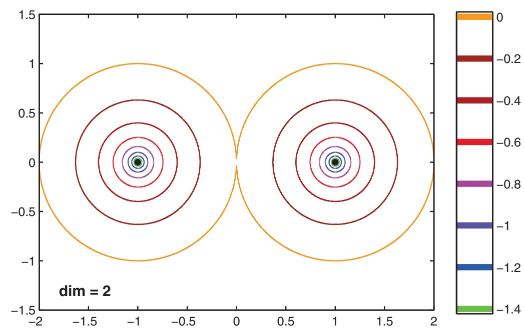
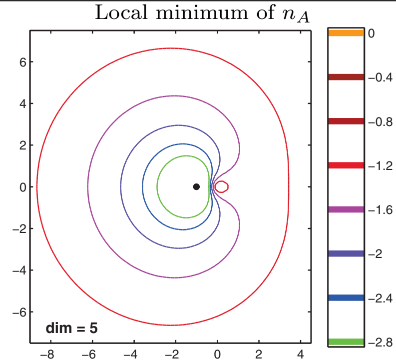
TABLE 3
Examples of pseudospectra for Example 4.10.

A	Diagram
$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	Smooth Saddle 
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Nonsmooth Saddle 
$-\begin{pmatrix} 1 & 5 & 5^2 & 5^3 & 5^4 \\ 0 & 1 & 5 & 5^2 & 5^3 \\ 0 & 0 & 1 & 5 & 5^2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	Local minimum of n_A 

We also have an alternative proof to [4, Theorem 9.2] after the remark below.

Remark 4.11. The set

$$G(z) = \left\{ v^H (A - zI)\, v \mid v \in V(z), \|v\| = 1 \right\},$$

where the subspace $V(z) \subset \mathbb{C}^n$ is spanned by all right singular vectors of $A - zI$ as defined in [4, Section 9], is equal to $\underline{\sigma}(A - zI)\, Y(A - zI)$.

PROPOSITION 4.12. *If $\bar{z}$ is not resolvent-critical and $\underline{\sigma}_A(\bar{z}) = \epsilon$, then the set $\Lambda_\epsilon^c(A)$ is Clarke regular at $\bar{z}$, with normal cone $N_{\Lambda_\epsilon^c(A)}(\bar{z}) = \text{pos}\,(Y(A - \bar{z}I))$.*

Proof. This involves applying Proposition 4.5 to a result on levels sets [21, Proposition 10.3]. $\square$

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The conditions below on $\partial\underline{\sigma}^e(A,z)$ and $\partial(-\underline{\sigma}^e)(A,z)$ are needed in a manner similar to condition (b) in Theorem 2.10 in the proof of our main result.

PROPOSITION 4.13. *The condition* $(0,0)\in\partial\underline{\sigma}^e(A,z)$ *holds if and only if* $z\in\Lambda(A)$. *Also, if* $z\notin\Lambda(A)$, *then* $(0,0)\notin\partial(-\underline{\sigma}^e)(A,z)$.

Proof. If $\underline{\sigma}^e(A,z)=0$, then (A,z) is a local minimizer, and thus $(0,0)\in\partial\underline{\sigma}^e(A,z)$. On the other hand, if $\underline{\sigma}^e(A,z)>0$, we need to prove that $(0,0)\notin\partial\underline{\sigma}^e(A,z)$. We try to evaluate $\hat{\partial}\underline{\sigma}^e(A,z)$. From Proposition 4.3, we know that at points where the multiplicity of the singular value $\underline{\sigma}(A-zI)$ is greater than one, $\underline{\sigma}^e$ is not differentiable. By [21, Corollary 9.21], $\hat{\partial}\underline{\sigma}^e(A,z)=\emptyset$ at these points. For points where the multiplicity of the singular value is one, the norm calculation tells us that the only point in $\hat{\partial}\underline{\sigma}^e(A,z)$ has norm at least one; the only element in $\hat{\partial}\underline{\sigma}^e(A,z)$ is of the form $\left(uv^H,-v^Hu\right)$, and the matrix part already contributes one to the norm. So it is impossible that $(0,0)\in\partial\underline{\sigma}^e(A,z)$.

Next, we move on to $\partial(-\underline{\sigma}^e)(A,z)$. Take $\bar{U},\bar{V}$ to be the matrix corresponding to the left and right singular vectors of $A-zI$ in the sense of Proposition 3.3. Note that $\left(\bar{U}\bar{V}^H,0\right)$ represents a direction of linear descent, as

$$-\underline{\sigma}^e\left(A+\epsilon\bar{U}\bar{V}^H,z\right)=-\underline{\sigma}^e(A,z)-\epsilon$$

for small ϵ, and so we have $(0,0)\notin\hat{\partial}(-\underline{\sigma}^e)(A,z)$. Due to regularity (Proposition 4.3), we have $(0,0)\notin\partial(-\underline{\sigma}^e)(A,z)$. $\square$

Despite the fact that $\underline{\sigma}^e$ is not regular, we are still able to calculate the subdifferential $\partial\underline{\sigma}^e(A,z)$.

PROPOSITION 4.14. *If* $z\notin\Lambda(A)$, *then*

$$\partial\underline{\sigma}^e(A,z)=\left\{\left(uv^H,-v^Hu\right)\mid(u,v)\in MSV(A-zI)\right\}.$$

Proof. We observe that

$$\begin{aligned}\partial\underline{\sigma}^e(A,z)&\subset-\partial(-\underline{\sigma}^e)(A,z)\\&=\operatorname{conv}\left\{\left(uv^H,-v^Hu\right)\mid(u,v)\in MSV(A-zI)\right\}\end{aligned}$$

by [21, Corollary 9.21] and Proposition 4.3. Next, note that if $(B,w)\in\partial\underline{\sigma}^e(A,z)$, then

$$(B,w)\in\operatorname{conv}\left\{\left(uv^H,-v^Hu\right)\mid(u,v)\in MSV(A-zI)\right\},$$

and so we may write $(B,w)=\sum_{i=1}^k\lambda_i\left(u_iv_i^H,-v_i^Hu_i\right)$ for a convex combination of left and right singular vectors u_i,v_i corresponding to the smallest singular value. But since the 2-norm is a strictly convex norm, $\|B\|<1$ if $k>1$ and (u_i,v_i)'s are not complex multiples each other. We take a closer look: (B,w) can be written as a limit of $(B_i,w_i)=\nabla\underline{\sigma}^e(A_i,z_i)$, where $(A_i,z_i)\to(A,z)$ by [21, Corollary 9.21]. Since $\|B_i\|=1$, it follows that $\|B\|=1$.

With this, we conclude that $(B,w)=(uv^*,-v^*u)$ for some $(u,v)\in MSV(A-zI)$, and so

$$\partial\underline{\sigma}^e(A,z)\subset\left\{\left(uv^H,-v^Hu\right)\mid(u,v)\in MSV(A-zI)\right\}.$$

To prove the other containment, note that, for any $(u,v)\in MSV(A-zI)$, we have

$$\begin{aligned}\hat{\partial}\underline{\sigma}^e\left(A-\delta uv^H,z\right)&=\left\{\nabla\underline{\sigma}^e\left(A-\delta uv^H,z\right)\right\}\\&=\left\{\left(uv^H,-v^Hu\right)\right\}\end{aligned}$$

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for $0 < \delta < \epsilon$ by Lemma 4.2. Taking limits as $\delta \downarrow 0$, we have $\left(uv^H, -v^H u\right) \in \partial \underline{\sigma}^e (A, z)$, which completes the proof. ∎

5. Main result. Before proving our main result, we make a statement about the normal cones $N_{\mathrm{gph}\,\Lambda_\epsilon^c}(A, z)$ and $N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z)$. We make use of properties that we have established in section 4 to establish the link between level sets and normal vectors.

PROPOSITION 5.1. *If $\epsilon = \underline{\sigma}^e (A, z) > 0$, then*

$$N_{\mathrm{gph}\,\Lambda_\epsilon^c}(A, z) = \mathrm{pos\ conv}\left\{\left(-uv^H, v^H u\right) \mid (u, v) \in MSV(A - zI)\right\},$$
$$N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z) = \mathrm{pos}\left\{\left(uv^H, -v^H u\right) \mid (u, v) \in MSV(A - zI)\right\}.$$

Proof. Apply a result on level sets [21, Proposition 10.3], Proposition 4.13, and the fact that $-\underline{\sigma}^e$ is Lipschitz to get

$$N_{\mathrm{gph}\,\Lambda_\epsilon^c}(A, z) = \mathrm{pos}\left(\partial\left(-\underline{\sigma}^e\right)(A, z)\right).$$

Next, apply Proposition 4.3 to deduce the first result.

By [21, Proposition 10.3] and Proposition 4.14, we have

$$\begin{aligned} N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z) &\subset \mathrm{pos}\ \partial \underline{\sigma}^e (A, z) \\ &= \mathrm{pos}\left\{\left(uv^H, -v^H u\right) \mid (u, v) \in MSV(A - zI)\right\}. \end{aligned}$$

Furthermore, if $\underline{\sigma}(A - zI)$ is simple, then $\underline{\sigma}^e$ is smooth and regular at (A, z) by Lemma 4.2, and so the above inclusion holds with equality.

For the opposite containment, take any $(u, v) \in MSV(A - zI)$. Consider the pair

$$(A_\delta, z_\delta) := \left((1 + \delta) A - \epsilon\delta uv^H, (1 + \delta) z\right) \text{ for small } \delta > 0.$$

At these points, $\underline{\sigma}^e$ is smooth (and thus regular) because the singular value is of multiplicity one with corresponding singular vectors (u, v), and $\underline{\sigma}^e (A_\delta, z_\delta) = \epsilon$. Thus

$$\left(uv^H, -v^H u\right) \in \hat{N}_{\mathrm{gph}\,\Lambda_\epsilon}\left((1 + \delta) A - \epsilon\delta uv^H, (1 + \delta) z\right).$$

Taking $\delta \downarrow 0$, we see that $\left(uv^H, -v^H u\right) \in N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z)$. Since $N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z)$ is a cone, we have the formula for $N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z)$ as claimed. ∎

The following is the main result that summarizes the links between Figure 1 in the introduction.

THEOREM 5.2. *Consider a point $z \notin \Lambda(A)$. Let $\epsilon = \underline{\sigma}^e (A, z)$. Then the following are equivalent:*

(1) *z is not resolvent-critical for A.*
(2) *Λ_ϵ^c has the Aubin property at A for z.*
(3) *Λ_ϵ has the Aubin property at A for z.*

Proof. For the purposes of the proof, we introduce several other properties:

(4) $(M^n \times \{0\}) \cap N_{\mathrm{gph}\,\Lambda_\epsilon^c}(A, z) = \{0\}$.
(5) $D^*\Lambda_\epsilon^c (A \mid z)(0) = \{0\}$.
(6) $(M^n \times \{0\}) \cap N_{\mathrm{gph}\,\Lambda_\epsilon}(A, z) = \{0\}$.
(7) $D^*\Lambda_\epsilon (A \mid z)(0) = \{0\}$.

Properties (4) and (5) are equivalent because $\alpha \in D^*\Lambda_\epsilon^c (A \mid z)(\beta)$ if and only if $(\alpha, -\beta) \in N_{\mathrm{gph}\,\Lambda_\epsilon^c}(A, z)$ by the definition of coderivatives [21, Definition 8.33].

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Properties (5) and (2) are equivalent by the Mordukhovich Criterion [21, Theorem 9.40]. The same goes for properties (6), (7), and (3).

Next, we show the equivalence of properties (1) and (4). We apply Proposition 5.1 to reduce property (4) to

$$(M^n \times \{0\}) \cap \text{pos conv}\left\{\left(-uv^H, v^H u\right) \mid (u,v) \in MSV(A - zI)\right\} = \{0\}.$$

$(1 \Rightarrow 4)$ Suppose that z is not resolvent-critical, that is, $0 \notin Y(A - zI)$, and yet property (4) fails. Then there is some nonzero pair with second coordinate (the one in $\mathbb{C}$) zero lying in

$$\text{pos conv}\left\{\left(-uv^H, v^H u\right) \mid (u,v) \in MSV(A - zI)\right\}.$$

This means that there is a convex combination of pairs $\left(-uv^H, v^H u\right)$ such that their second coordinate is zero. Then $0 \in Y(A - zI)$ (appealing to Proposition 3.4), a contradiction.

$(1 \Leftarrow 4)$ If property (1) fails, there are minimal left and right singular vectors u, v such that $v^H u = 0$, and then $\left(-uv^H, v^H u\right)$ is a nonzero element in

$$(M^n \times \{0\}) \cap \text{pos conv}\left\{\left(-uv^H, v^H u\right) \mid (u,v) \in MSV(A - zI)\right\}.$$

So we have proved the equivalence of properties (1) and (4). We proceed to prove the equivalence of properties (1) and (6). We lose regularity, but nevertheless, the proof still looks similar.

$(1 \Rightarrow 6)$ We prove $(4 \Rightarrow 6)$. If $0 \notin Y(A - zI)$, then $(M^n \times \{0\}) \cap N_{\text{gph}\,\Lambda_\epsilon^c}(A, z) = \{0\}$. But Proposition 5.1 gives

$$\begin{aligned}\{0\} &\subset (M^n \times \{0\}) \cap N_{\text{gph}\,\Lambda_\epsilon}(A, z)\\ &\subset (M^n \times \{0\}) \cap -N_{\text{gph}\,\Lambda_\epsilon^c}(A, z)\\ &= \{0\}.\end{aligned}$$

$(1 \Leftarrow 6)$. If property (1) fails, there are minimal left and right singular vectors u, v such that $v^H u = 0$, and thus $\left(uv^H, -v^H u\right)$ is a nonzero element in $(M^n \times \{0\}) \cap N_{\text{gph}\,\Lambda_\epsilon}(A, z)$. ∎

When we consider fixing the matrix A and increasing ϵ, it is natural to ask whether the map $\epsilon \mapsto \Lambda_\epsilon(A)$ is Lipschitz.

Proposition 5.3. *Given $z \in \mathbb{C}$, the map $\epsilon \mapsto \Lambda_\epsilon(A)$ has the Aubin property at $\underline{\sigma}_A(z)$ for z if and only if $0 \notin \partial\underline{\sigma}_A(z)$, whereas the map $\epsilon \mapsto \Lambda_\epsilon^c(A)$ has the Aubin property at $\underline{\sigma}_A(z)$ for z if and only if $0 \notin \partial(-\underline{\sigma}_A)(z)$ (or equivalently, assuming $z \notin \Lambda(A)$, z is not resolvent-critical for A).*

Proof. A straightforward application of [21, Theorem 9.41(b)] on $\underline{\sigma}_A$ gives us $0 \notin \partial\underline{\sigma}_A(z)$ if and only if the map $\epsilon \mapsto \text{lev}_{\leq\epsilon}\underline{\sigma}_A = \Lambda_\epsilon(A)$ has the Aubin property at ϵ for z, which is the first part of what we seek to prove. The second part is similar, using Proposition 4.5. ∎

A particular example worked out in full detail exploiting this is highlighted in [6].

It is natural to ask whether there are any differences between Theorem 5.2 and the two parts of Proposition 5.3, and it comes down to comparing $\partial(-\underline{\sigma}_A)$ and $\partial\underline{\sigma}_A$. In general, if z is not an eigenvalue of A,

$$-\partial\underline{\sigma}_A(z) \subset \partial(-\underline{\sigma}_A)(z) = Y(A - zI)$$

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by Proposition 4.5 and [21, Corollary 9.21], but the inclusion can be strict. Consider the matrix $\bar{A} = \operatorname{diag}(1, -1, i, -i)$ in Example 2.6. Here,

$$\partial\left(-\underline{\sigma}_A\right)(0) = \{a + bi \mid |a| + |b| \leq 1\},$$

so 0 is resolvent-critical while $\partial \underline{\sigma}_A(0) = \{1, -1, i, -i\}$.

6. Lipschitz continuity of pseudospectra. The results in the last section study the Aubin property of the pseudospectra Λ_ϵ. The next natural step is to evaluate the graphical modulus and investigate the Lipschitz continuity of Λ_ϵ.

If $\underline{\sigma}(A - zI) = \epsilon > 0$, then from Proposition 5.1 and the definition of the coderivative, we can deduce the formula for $D^*\Lambda_\epsilon^c(A \mid z)(c)$. To keep the expressions compact, we understand that (u_i, v_i) ranges over $MSV(A - zI)$ whenever u_i, v_i appear in the formulas below. We have

$$\begin{aligned}
&D^*\Lambda_\epsilon^c(A \mid z)(c)\\
&= \left\{-k\sum_i \lambda_i u_i v_i^H \mid c = -k\sum_i \lambda_i v_i^H u_i, \sum_i \lambda_i = 1, \lambda_i \geq 0, k \geq 0\right\}\\
&= \begin{cases} \left\{c\frac{\sum_i \lambda_i u_i v_i^H}{\sum_i \lambda_i v_i^H u_i} \mid \sum_i \lambda_i v_i^H u_i \neq 0\right\} & \text{if } c \neq 0,\\ \operatorname{pos}\left\{\sum_i \lambda_i u_i v_i^H \mid \sum_i \lambda_i v_i^H u_i = 0\right\} & \text{if } c = 0,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&D^*\Lambda_\epsilon(A \mid z)(c)\\
&= \left\{kuv^H \mid c = kv^H u, k \geq 0, (u, v) \in MSV(A - zI)\right\}\\
&= \begin{cases} \left\{c\frac{uv^H}{v^H u} \mid (u, v) \in MSV(A - zI), v^H u \neq 0\right\} & \text{if } c \neq 0,\\ \operatorname{pos}\left\{uv^H \mid (u, v) \in MSV(A - zI), v^H u = 0\right\} & \text{if } c = 0.\end{cases}
\end{aligned}$$

We can then calculate the graphical moduli for Λ_ϵ and Λ_ϵ^c in the theorem below.

THEOREM 6.1. *We have the following graphical moduli:*

$$\operatorname{lip}\Lambda_\epsilon(A \mid z) = \begin{cases} 1/d(0, Y(A - zI)) & \textit{if } \underline{\sigma}(A - zI) = \epsilon,\\ 0 & \textit{if } \underline{\sigma}(A - zI) < \epsilon,\end{cases}$$

$$\operatorname{lip}\Lambda_\epsilon^c(A \mid z) = \begin{cases} 1/d(0, Y(A - zI)) & \textit{if } \underline{\sigma}(A - zI) = \epsilon,\\ 0 & \textit{if } \underline{\sigma}(A - zI) > \epsilon.\end{cases}$$

(Here, we interpret $1/0 = +\infty$.*)*

Proof. It is clear that if $\underline{\sigma}(A - zI) < \epsilon$, then (A, z) lies in the interior of $\operatorname{gph}\Lambda_\epsilon$, so $N_{\operatorname{gph}\Lambda_\epsilon}(A, z) = \{(0, 0)\}$, and so

$$\operatorname{lip}\Lambda_\epsilon(A \mid z) = |D^*\Lambda_\epsilon(A \mid z)|^+ = 0.$$

Similarly, $\operatorname{lip}\Lambda_\epsilon^c(A \mid z) = 0$ if $\underline{\sigma}(A - zI) > \epsilon$.

If $\underline{\sigma}(A - zI) = \epsilon$ and $0 \in Y(A - zI)$, then Λ_ϵ and Λ_ϵ^c do not have the Aubin property at A for z, and so

$$\operatorname{lip}\Lambda_\epsilon(A \mid z) = \operatorname{lip}\Lambda_\epsilon^c(A \mid z) = \infty.$$

By the Mordukhovich criterion [21, Theorem 9.40] and the definition of outer norms [21, Section 9D], we have $\operatorname{lip}\Lambda_\epsilon^c(A \mid z)$ to be

$$\sup_{c \neq 0} \sup_{d \in D^*\Lambda_\epsilon^c(A|z)(c)} \frac{\|d\|}{|c|},$$

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or, in other words, the infimum of all κ such that

$$d \in D^* \Lambda_\epsilon^c (A \mid z)(c) \implies \|d\| \leq \kappa |c|. \tag{6.1}$$

In view of the formula for $D^* \Lambda_\epsilon^c (A \mid z)$, formula (6.1) is equivalent to

$$\left\| \sum \lambda_i u_i v_i^H \right\| \leq \kappa \left| \sum \lambda_i v_i^H u_i \right| \tag{6.2}$$

for all $(u_i, v_i) \in MSV(A - zI), \lambda_i \geq 0, \sum \lambda_i = 1$. To prove that $\operatorname{lip} \Lambda_\epsilon^c (A \mid z) = 1/d(0, Y(A - zI))$, it remains to prove that formula (6.2) is equivalent to

$$\kappa \geq 1/d(0, Y(A - zI)). \tag{6.3}$$

Suppose that κ satisfies formula (6.2). Then for $y \in Y(A - zI)$, we have some $(u, v) \in MSV(A - zI)$ such that $y = v^H u$. Then

$$\begin{aligned} \kappa |y| &= \kappa \left| v^H u \right| \\ &\geq \left\| u v^H \right\| \\ &= 1. \end{aligned}$$

Formula (6.3) follows. Next, suppose that κ satisfies formula (6.3). If $(u_i, v_i) \in MSV(A - zI)$, $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, we have $\sum \lambda_i v_i^H u_i \in Y(A - zI)$ by the convexity of $Y(A - zI)$. Thus

$$\begin{aligned} \left\| \sum \lambda_i u_i v_i^H \right\| &\leq \sum \lambda_i \left\| u_i v_i^H \right\| \\ &= 1 \\ &\leq \kappa \left| \sum \lambda_i v_i^H u_i \right|. \end{aligned}$$

Formula (6.2) follows, and so $\operatorname{lip} \Lambda_\epsilon^c (A \mid z) = 1/d(0, Y(A - zI))$. Similar and simpler calculations give us $\operatorname{lip} \Lambda_\epsilon (A \mid z) = 1/d(0, Y(A - zI))$. □

We next turn to the Lipschitz constant for the pseudospectral mapping Λ_ϵ. We want to find $\operatorname{lip}_\infty \Lambda_\epsilon (\bar{A})$, the Lipschitz modulus of the pseudospectral map at $\bar{A}$. For a set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we are able to calculate $\operatorname{lip}_\infty S(\bar{x})$ from the graphical modulus easily with the following formula.

Proposition 6.2 (see [20, Theorem 1.42]). *If $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous and S is locally bounded at $\bar{x}$, then*

$$\operatorname{lip}_\infty S(\bar{x}) = \max_{y \in S(\bar{x})} \operatorname{lip} S(\bar{x} \mid y).$$

Thus the Lipschitz constants for Λ_ϵ are easily obtained.

Proposition 6.3. *The following expressions are equal:*

(i) $\operatorname{lip}_\infty \Lambda_\epsilon (A)$,

(ii) $\max\limits_{z \in \Lambda_\epsilon(A)} \{\operatorname{lip} \Lambda_\epsilon (A \mid z)\}$,

(iii) $\max\limits_{z : \underline{\sigma}(A - zI) = \epsilon} \{1/d(0, Y(A - zI))\}$,

(iv) $\max\limits_{z} \left\{ 1/\left| v^H u \right| \mid (u, v) \in MSV(A - zI), \underline{\sigma}(A - zI) = \epsilon \right\}$.

Proof. The expressions (i) and (ii) are equal by Proposition 6.2 and the fact that Λ_ϵ is compact and locally bounded. Then expressions (ii) and (iii) are equal by Theorem 6.1, and expression (iv) is just an expansion of the definition of $Y(\cdot)$ applied to expression (iii). □

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7. Pseudospectral abscissa and pseudospectral radius. In this section we apply our results on Lipschitz continuity of pseudospectra to reexplore earlier work on the pseudospectral abscissa and pseudospectral radius in [4, 5, 19, 22].

DEFINITION 7.1. *Define the* ϵ-pseudospectral abscissa $\alpha_\epsilon : M^n \to \mathbb{R}$ *by*

$$\alpha_\epsilon(A) = \max_{z \in \Lambda_\epsilon(A)} \operatorname{Re}(z),$$

and the ϵ-pseudospectral radius $\rho_\epsilon : M^n \to \mathbb{R}_+$ *by*

$$\rho_\epsilon(A) = \max_{z \in \Lambda_\epsilon(A)} |z|.$$

Note that if $\epsilon > 0$, then $\rho_\epsilon(A) > 0$. We shall establish the continuity properties of α_ϵ and ρ_ϵ. We begin with another routine piece of theory on parametric minimization.

COROLLARY 7.2 (to [21, Corollary 10.14]). *Suppose that* $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and maps to compact sets. Define* $p : \mathbb{R}^m \to \mathbb{R}$ *and* $P : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *below by*

$$p(u) = \min_{x \in F(u)} g(x), \quad P(u) = \arg\min_{x \in F(u)} g(x),$$

where the lower semicontinuous function $g : \mathbb{R}^n \to \mathbb{R}$ *is differentiable at all points in* $P(\bar{u})$ *for some given* $\bar{u} \in \mathbb{R}^m$. *Then* p *is*

(a) *Lipschitz continuous around* $\bar{u}$ *if* F *has the Aubin property at* $\bar{u}$ *for* $\bar{x}$ *for all* $\bar{x} \in P(\bar{u})$, *with*

$$\operatorname{lip} p(\bar{u}) \le \max\{|y| : y \in S\} < \infty,$$

where $S = \{y \mid \bar{x} \in P(\bar{u}), y \in D^*F(\bar{u} \mid \bar{x})(\nabla g(\bar{x}))\}$;

(b) *strictly differentiable at* $\bar{u}$ *with* $\nabla p(\bar{u}) = \bar{y}$ *if* $S = \{\bar{y}\}$.

Proof. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ be defined by

$$f(x,u) = \delta_{\operatorname{gph} F}(u,x) + g(x) = \begin{cases} g(x) & \text{if } x \in F(u), \\ \infty & \text{otherwise.} \end{cases}$$

Then

$$p(u) = \inf_x f(x,u), \quad P(u) = \arg\min_x f(x,u).$$

Since F is outer semicontinuous, $\operatorname{gph} F$ is closed, so f is proper and lower semicontinuous.

Next, we prove f is level bounded in x locally uniformly in u. That is, for each $\bar{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, there is a neighborhood V of $\bar{u}$ along with a bounded set $B \subset \mathbb{R}^n$ such that $\{x \mid f(x,u) \le \alpha\} \subset B$ for all $u \in V$. Note that $f(x,u) \le \alpha$ means that $x \in F(u)$ and $g(x) \le \alpha$. Since F is outer semicontinuous, choose V such that $F(u) \subset F(\bar{u}) + \mathbb{B}$ for all $u \in V$, by the characterization of outer semicontinuity in [21, Proposition 5.12]. The set B can be chosen to be $F(\bar{u}) + \mathbb{B}$, and we are done.

Following the notation in [21, Corollary 10.13], for any $\bar{x} \in P(\bar{u})$,

$$\begin{aligned} M(\bar{x},\bar{u}) &:= \{y \mid (0,y) \in \partial f(\bar{x},\bar{u})\} \\ &= \{y \mid (y,0) \in \partial \delta_{\operatorname{gph} F}(\bar{u},\bar{x}) + \{(0,\nabla g(\bar{x}))\}\} \\ &\qquad \text{(by [21, Exercise 8.8(c)])} \\ &= \{y \mid (y, -\nabla g(\bar{x})) \in N_{\operatorname{gph} F}(\bar{u},\bar{x})\} \\ &\qquad \text{(by [21, Exercise 8.14])} \\ &= D^*F(\bar{u} \mid \bar{x})(\nabla g(\bar{x})) \\ &\qquad \text{(by [21, Definition 8.33]).} \end{aligned}$$

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Also,

$$\begin{aligned} M_\infty(\bar{x},\bar{u}) &:= \{y \mid (0,y) \in \partial^\infty f(\bar{x},\bar{u})\} \\ &= \{y \mid (y,0) \in \partial^\infty \delta_{\mathrm{gph}F}(\bar{u},\bar{x})\} \\ &= \{y \mid (y,0) \in N_{\mathrm{gph}\,F}(\bar{u},\bar{x})\} \\ &= D^*F(\bar{u} \mid \bar{x})(0). \end{aligned}$$

This means that $Y_\infty(\bar{u}) := \bigcup_{\bar{x}\in P(\bar{u})} M_\infty(\bar{x},\bar{u}) = \{0\}$, so part (a) of [21, Corollary 10.14] applies. Furthermore, $Y(\bar{u})$, where $Y(\cdot)$ is defined in [21, Corollary 10.13], is

$$\begin{aligned} Y(\bar{u}) &:= \bigcup_{\bar{x}\in P(\bar{u})} M(\bar{x},\bar{u}) \\ &= \bigcup_{\bar{x}\in P(\bar{u})} D^*F(\bar{u} \mid \bar{x})(\nabla g(\bar{x})), \end{aligned}$$

and so

$$\begin{aligned} \mathrm{lip}\ p(\bar{u}) &\le \max_{y\in Y(\bar{u})} |y| \\ &= \max\{|y| : \bar{x} \in P(\bar{u}), y \in D^*F(\bar{u} \mid \bar{x})(\nabla g(\bar{x}))\} < \infty. \end{aligned}$$

The rest of the claim follows by [21, Corollary 10.14]. $\square$

The continuity of α_ϵ and ρ_ϵ can be proved by the following proposition when the conditions for Lipschitz continuity are absent. The proof is routine.

PROPOSITION 7.3. *Suppose that $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is continuous and maps to compact sets. If p, P, and g are defined as in Corollary* 7.2 *with g continuous, then p is continuous and P is outer semicontinuous.*

As a consequence of Corollary 7.2, we obtain the following result.

COROLLARY 7.4. *The pseudospectral abscissa α_ϵ and pseudospectral radius ρ_ϵ are Lipschitz continuous at a matrix A if $\mathrm{lip}_\infty \Lambda_\epsilon(A) < \infty$, with Lipschitz constants bounded above by $\mathrm{lip}_\infty \Lambda_\epsilon(A)$.*

Proof. Following the notation in Corollary 7.2, take $F = \Lambda_\epsilon$ and $g(x) = \langle -1, x\rangle$. Then $\alpha_\epsilon = -p$, and we obtain

$$\begin{aligned} \mathrm{lip}\ \alpha_\epsilon(A) &\le \max\{|y| : y \in D^*\Lambda_\epsilon(A \mid z)(-1) \\ &\qquad , z \in \Lambda_\epsilon(A), \mathrm{Re}(z) = \alpha_\epsilon(A)\} \\ &= \max\{1/d(0, \mathbb{R}_- \cap Y(A - zI)) : \\ &\qquad z \in \Lambda_\epsilon(A), \mathrm{Re}(z) = \alpha_\epsilon(A)\} \end{aligned}$$

using our derivative computation before Theorem 6.1. If we take $g(x) = -|x|$ instead, then $\rho_\epsilon = -p$, and

$$\begin{aligned} \mathrm{lip}\ \rho_\epsilon(A) &\le \max\left\{|y| : y \in D^*\Lambda_\epsilon(A \mid z)\left(-\frac{z}{|z|}\right),\right. \\ &\qquad \left. z \in \Lambda_\epsilon(A), |z| = \rho_\epsilon(A)\right\} \\ &= \max\left\{1/d\left(0, \mathbb{R}_+\left(\frac{z}{|z|}\right) \cap Y(A - zI)\right) :\right. \\ &\qquad \left. z \in \Lambda_\epsilon(A), |z| = \rho_\epsilon(A)\right\}. \end{aligned}$$

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The upper bounds for lip $\alpha_\epsilon(A)$ and lip $\rho_\epsilon(A)$ obtained above are both not greater than $\text{lip}_\infty \Lambda_\epsilon(A)$ by Proposition 6.3, and so we are done. □

8. Resolvent-critical points. Resolvent-critical points are crucial throughout our analysis. They are also, for example, explicitly excluded in the analysis of the quadratic convergence of the algorithm for finding the pseudospectral abscissa in [5]. We investigate their properties further.

PROPOSITION 8.1. *All resolvent-critical points lie in the numerical range of A.*

Proof. Suppose that z is resolvent-critical. Then there exists a right singular vector v of $(A - zI)$ such that $v^H(A - zI)v = 0$, which implies that $v^H Av = zv^H v = z$ if $|v| = 1$. This means that z lies in the numerical range of A. □

PROPOSITION 8.2. *For ϵ large enough such that $\Lambda_\epsilon(A)$ contains the numerical range of A, $W(A)$, in its interior, the map $\Lambda_\epsilon : M^n \rightrightarrows \mathbb{C}$ is strictly continuous at A for any point in $\Lambda_\epsilon(A)$, and thus Lipschitz continuous at a neighborhood of A. For α_ϵ and ρ_ϵ to be Lipschitz continuous, we just need the interior of* $\text{conv}\, \Lambda_\epsilon(\text{A})$ *to contain $W(A)$.*

Proof. For the first part, if $\Lambda_\epsilon(A)$ contains $W(A)$ in its interior, then the points in the boundary of Λ_ϵ are not resolvent-critical by the previous result. Apply Proposition 6.3.

For the second part, by the proof of Corollary 7.4, it suffices to show that if z satisfies $\text{Re}\, z = \alpha_\epsilon(A)$ and $\underline{\sigma}(A - zI) = \epsilon$, then $z \notin W(A)$. But if z satisfies these conditions, then $z \in \text{conv}\, \Lambda_\epsilon(A)$. The same goes for ρ_ϵ. □

Remark 8.3. In Table 3 in page 1061, the third example of a 5×5 matrix illustrates that a resolvent-critical can lie outside the convex hull of the spectrum of A. There is a resolvent-critical point close to 0, but the convex hull of the eigenvalues is just $\{-1\}$.

With all that we have done so far, the following is a natural consequence of [3, Corollary 8].

COROLLARY 8.4 (to [3, Corollary 8]). *Given a matrix A, the set of resolvent-critical values $\{\underline{\sigma}_A(z) \mid z$ resolvent critical for $A\}$ is finite.*

Proof. This is just the (semialgebraic) set of Clarke-critical values of $\underline{\sigma}_A$ by Theorem 4.9, which is finite by [3, Corollary 8]. □

With the above result, we arrive at the following appealing result.

COROLLARY 8.5. *Given a matrix A, the mappings Λ_ϵ, α_ϵ, and ρ_ϵ are Lipschitz around A for all but finitely many $\epsilon \geq 0$, so, in particular, for all small $\epsilon > 0$.*

Proof. This is a direct consequence of Theorem 5.2 and Corollaries 8.4 and 7.4. □

Remark 8.6. The conditions that guarantee Lipschitz continuity of the pseudospectral abscissa α_ϵ in the result above are much more general than the conditions in [4, Corollary 8.3]. Firstly, we do not need the assumption that active eigenvalues are nonderogatory made in [4, Corollary 8.3], and our current result holds for all but finitely many ϵ.

Here is another general observation on resolvent-critical points.

THEOREM 8.7. *For a fixed A, the set of resolvent-critical points is compact, semialgebraic with empty interior, and contains eigenvalues as isolated points.*

Proof. Denote the set of resolvent-critical points by S_A. The set S_A is bounded by Proposition 8.1. It is clear that S_A is semialgebraic. As $\underline{\sigma}_A$ is Lipschitz, $\partial^\circ(-\underline{\sigma}_A)$ has closed graph by [9, Proposition 2.1.5(b)], and thus S_A is closed.

Suppose that S_A does not have empty interior. Note that $\underline{\sigma}_A$ has to be constant on a component by Corollary 8.4, and this would mean that $\underline{\sigma}_A$ is constant on a set of

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nonempty interior, which contradicts the fact that $\underline{\sigma}_A$ cannot have minimizers other than at the eigenvalues of A [4, Theorem 4.2]. Thus S_A has empty interior.

Lastly, S_A can be written as a union of curves and points in $\mathbb{C}$. If an eigenvalue, say $\bar{z}$, is not an isolated point in S_A, then it is on some curve. This would mean that $\underline{\sigma}_A$ is zero on a curve, which contradicts the fact that $\underline{\sigma}_A$ is zero only on the set of eigenvalues, which is a finite set. Thus all eigenvalues are isolated in S_A. ☐

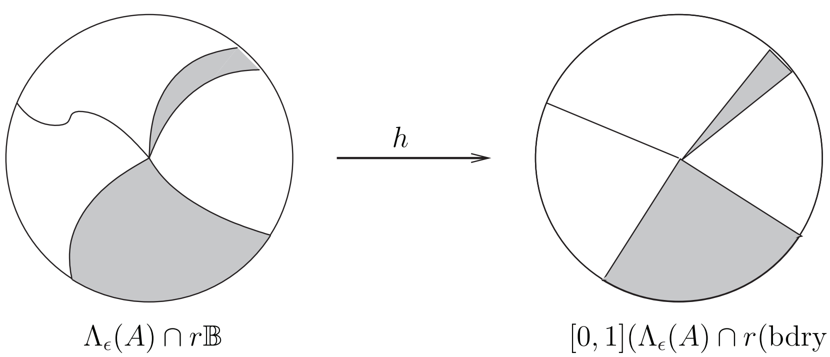
We call $\Lambda'_\epsilon(A) = \{z \mid \underline{\sigma}(A - zI) < \epsilon\}$ the *strict pseudospectrum* of A. The set $\Lambda'_\epsilon(A)$ consists of at most n components (since each component must contain an eigenvalue [22]), and the number of components is clearly a decreasing function of ϵ. There will be some points $\bar{z} \in \mathbb{C}$ where some components meet as ϵ increases. If $\Lambda'_\epsilon(A)$ has n components and $\bar{z}$ lies on the boundary of two components of $\Lambda'_\epsilon(A)$, then the distance between A and the set of matrices with repeated eigenvalues is ϵ and is attained by some matrix $\bar{A}$ having $\bar{z}$ as a repeated eigenvalue (see [1, Theorem 5.1]): It turns out that such points are resolvent-critical as the next theorem will show, generalizing [1, Proposition 4.10].

THEOREM 8.8. *If $\bar{z}$ is a common boundary point of two or more distinct components of $\Lambda'_\epsilon(A)$, then $\bar{z}$ is a resolvent-critical point.*

Proof. To reduce notation, let us assume that $\bar{z} = 0$. The rest of the proof will follow by a translation. We look at the structure of $\Lambda_\epsilon(A)$ around 0, where $\epsilon > 0$. Since $\Lambda_\epsilon(A)$ is semialgebraic, $\Lambda_\epsilon(A)$ is locally conic about 0 by [11, Theorem 4.10]. That is, there is an $r > 0$ and a semialgebraic homeomorphism

$$h : \Lambda_\epsilon(A) \cap r\mathbb{B} \to [0,1]\left(\Lambda_\epsilon(A) \cap r\,(\mathrm{bdry}\,\mathbb{B})\right)$$

between the two spaces. Since $(\Lambda_\epsilon(A) \cap r\,(\mathrm{bdry}\,\mathbb{B}))$ is a finite union of arcs, it follows that the boundary of $\Lambda_\epsilon(A) \cap r\mathbb{B}$ would consist of curves which start from 0 and end at somewhere on $r\,(\mathrm{bdry}\,\mathbb{B})$. The diagram below illustrates this.



We use a proof by contradiction. Suppose that 0 is not resolvent-critical. Then $0 \notin Y(A)$, and by Proposition 4.12, $\Lambda^c_\epsilon(A)$ is Clarke regular at 0, with normal cone $N_{\Lambda^c_\epsilon(A)}(0) = \mathbb{R}_+ Y(A)$. Note that $N_{\Lambda^c_\epsilon(A)}(0)$ is pointed, otherwise $0 \in Y(A)$, contradicting the assumption that 0 is not resolvent-critical.

The set $\{z \mid \underline{\sigma}(A - zI) = \epsilon\}$ is semialgebraic and has empty interior since the only local minimizers of $\underline{\sigma}_A$ are eigenvalues of A [4, Theorem 4.2], and so it is a union of smooth curves. We now prove that the curves are boundaries of both $\Lambda_\epsilon(A)$ and $\Lambda^c_\epsilon(A)$. By considering the sign of $\underline{\sigma}_A - \epsilon$ on either side of such a curve, we distinguish three cases. In the following diagram, both Case 1 and Case 2 cannot hold, because the local maxima and local minima of $\underline{\sigma}_A$ are resolvent-critical, and this would make 0

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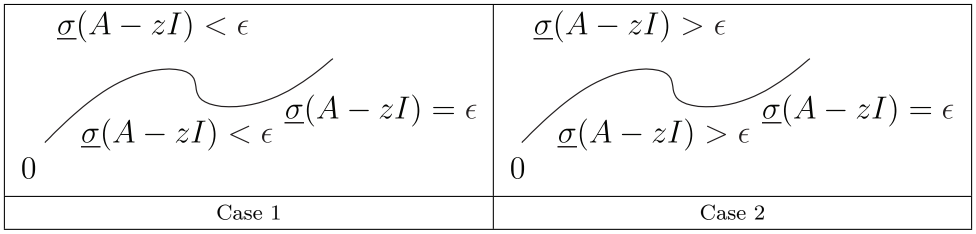
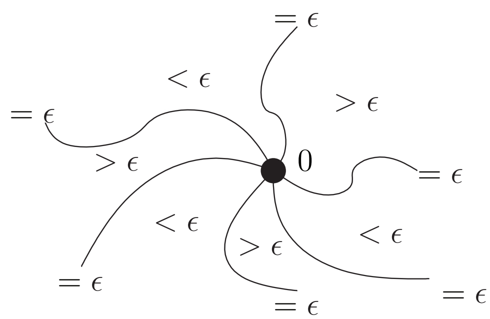
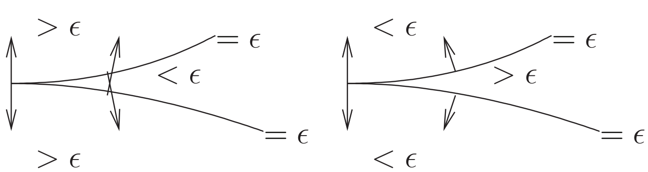
resolvent-critical as well, since the set of resolvent-critical points is closed by Theorem 8.7.



Therefore, the general diagram would be as below, with the value of $\underline{\sigma}_A$ alternating above and below ϵ as we circle the origin, crossing the curves where $\underline{\sigma}_A = \epsilon$.



Two different arcs cannot be tangent at 0 since $N_{\Lambda_\epsilon^c(A)}(0)$ will otherwise not be pointed, as the diagrams below show.



Since $\Lambda_\epsilon^c(A)$ is Clarke regular at 0, its tangent cone $T_{\Lambda_\epsilon^c(A)}(0)$ is convex, so the picture above can contain only one sector where $\underline{\sigma}_A > \epsilon$. It now follows that 0 cannot be the boundary point of two components of $\Lambda'_\epsilon(A)$. This completes the proof. ∎

If we can prove the following about the pseudospectral abscissa α_ϵ, then we can conclude that the pseudospectral abscissa is Lipschitz continuous.

CONJECTURE 8.9. *The points where the pseudospectral abscissa α_ϵ are attained are not resolvent-critical.*

A natural question to ask after Theorem 8.7 is the following.

CONJECTURE 8.10. *The number of resolvent-critical points is finite.*

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